Wellfare

Tutorial 2 Solutions

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1 Question 1

Either prove or disprove each of the following statements in the context of 2 × 2 games (disproving is usually best done with a counterexample):

a **If a player i has a dominant strategy in a game, then in every Nash equilibrium of that game player i will choose a dominant strategy.**

The statement is false. Consider the following game:

		Player II	
		L	R
Player I	T	0, 0	2, 1
	B	3, 2	1, 2

R is a weakly dominant strategy, but the game has two Nash equilibria: (T,R) and (B,L). The second of which does not contain any dominant strategies.

The following statements instead, are however, true. A Nash Equilibrium will never include a strictly dominated strategy and every dominant strategy equilibrium is a Nash equilibrium. If the game has a strictly dominant strategy equilibrium, then it is unique and it is also the Nash equilibrium.

b **If a game has a dominant strategy equilibrium, then it is unique: the game has no other dominant strategy equilibria.**

I'll accept both True and False based on how you interpret the question and the definition of dominant strategy equilibrium.

Definition 1 (from lecture slides) A dominant strategy equilibrium is a strategy profile in which every player has chosen a dominant strategy.

If we restrict to strictly dominant strategies, then the equilibrium must be unique and the game cannot have any other dominant strategy equilibrium. The proof is quite simple.

Assume there exists a second strictly dominant strategy equilibrium $\boldsymbol{\sigma}_2^* \neq \boldsymbol{\sigma}_1^*$. Then for some player i, we have $s_i \in \boldsymbol{\sigma}_1^*$ and $t_i \in \boldsymbol{\sigma}_2^*$ and $s_i \neq t_i$. However, if $\boldsymbol{\sigma}_1^*$ is a strictly dominant equilibrium, then for any $\boldsymbol{\sigma}_{-i} \in \boldsymbol{\Sigma}_{-i}$, we have $u(s_i, \boldsymbol{\sigma}_{-i}) > u(t_i, \boldsymbol{\sigma}_{-i})$ as s_i is strictly dominant. But then $\boldsymbol{\sigma}_2^*$ cannot be a strictly dominant strategy as t_i is dominated by s_i.

Now, even if we restrict to weakly dominant strategies, the claim is still true. Why? It comes from the following theorem (see supplementary notes for proof).

Theorem 1 Every player i can have only 1 weakly dominant strategy.

Now assume there is more than one weakly dominant equilibrium $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. If they were not the same, then there has to be a player i such that the two profiles differ in actions σ_1 and σ_2 for player i. However, both cannot be weakly dominant strategies, so we have a contradiction.

However, if we use the weakly dominant strategies definition from [Gin14, Exercise 4.2], then the statement is **false**.

Definition 2 (Princeton Definition Of Weakly Dominant Strategies) An action σ_i for player i is weakly dominant if the for all other actions $\sigma_i' \in \Sigma_i$, and any $\boldsymbol{\sigma}_{-i}$

$$u_i(\sigma_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\sigma_i', \boldsymbol{\sigma}_{-i}) \tag{1}$$

The second condition where it must sometimes be strictly better is not required. Under this definition, the following game works.

		Player II	
		Football	Tennis
Player I	Football	1, 1	1, 1
	Tennis	1, 1	1, 1

In the above game, all 4 actions are dominant strategy equilibrium.

c **Every dominant strategy equilibrium of a game is a Nash equilibrium.**

True. Assume that the statement is not true and that $\boldsymbol{\sigma}$ is a dominant strategy equilibrium and $\boldsymbol{\Sigma}^*$ be the set of Nash equilibria and $\boldsymbol{\sigma} \notin \boldsymbol{\Sigma}^*$. This implies that there exists a player i, who will choose to deviate from the $\boldsymbol{\sigma}$ strategy profile because they can improve their utility. However, for $\boldsymbol{\sigma}$ to be a dominant strategy, players i's actions $\boldsymbol{\sigma}[i]$ must be at least weakly dominant. This implies there cannot be another strategy $s_i \in \Sigma_i$, such that, for any $\boldsymbol{\sigma}_{-i}$ we have $u(s_i, \boldsymbol{\sigma}_{-i}) > u(\boldsymbol{\sigma}[i], \boldsymbol{\sigma}_{-i})$. This implies player i will not move. Hence, we have a contradiction and $\boldsymbol{\sigma}$ must be in $\boldsymbol{\Sigma}^*$.

d **Every Nash equilibrium of a game is a dominant strategy equilibrium.** False. See part a, (B,L) is a Nash equilibrium but it is not dominant strategy equilibrium.

e **If a game outcome $\boldsymbol{\sigma}^*$ maximises utilitarian social welfare, then $\boldsymbol{\sigma}^*$ is Pareto efficient.**

Assume that $\sum_{i \in N} u_i(\boldsymbol{\sigma}^*)$ is maximal. Now pick arbitrary i, who decides to swap σ_i^* with some other action s_i to get utilty $u_i(s_i, \boldsymbol{\sigma}^*_{-i}) > u_i(\sigma_i^*, \boldsymbol{\sigma}^*_{-i})$. As $\sum_{i \in N} u_i(\boldsymbol{\sigma}^*)$ is maximal, this implies there cannot be a $j \in N$ such that $u_j(s_i, \boldsymbol{\sigma}^*_{-i})$ remains unchanged or improves. Thus $\boldsymbol{\sigma}^*$ is pareto optimal.

f **If a game outcome $\boldsymbol{\sigma}^*$ is Pareto efficient, then it maximises utilitarian social welfare.**

		Player II	
		A	B
Player I	A	1, 1	1, 1
	B	1, 100	2, 1

Here clearly, (B,B) is the pareto efficient strategy but clearly (B,A) maximises wellfare. The intuition is that pareto optimal only says you cannot improve your state without harming others. It doesn't say anything about how much you improve vs how much you harm others.

g **If all utilities in a game are positive, then any outcome that maximises the product of utilities of players is Pareto efficient.**

log is a monotonic function. Thus, $\log(u_i)$ is also a utility function for the preferences of player i (See supplementary material for tutorial 1, specifically Theorem 1).

$$\log\left(\prod_{i\in N} u_i(\boldsymbol{\sigma}^*)\right) = \sum_{i\in N} \log u_i(\boldsymbol{\sigma}^*) \tag{2}$$

Define $\hat{u}_i(\boldsymbol{\sigma}) = \log u_i(\boldsymbol{\sigma})$ as the new utility function, and by assumption $\boldsymbol{\sigma}^*$ maximises aggregate utility. So, by part e, $\boldsymbol{\sigma}^*$ is pareto optimal.

h **If all utilities in a game are positive, then any Pareto efficient outcome of the game will maximise the product of utilities of players.**

False. See part (f). Apply log of product of utilities like in part (g). Answer follows.

2 Question 2

If we use mixed strategies in a game, then we are in the domain of expected utility.

a **Write down an expression for the expected utility of each player in a generic 2 × 2 game when a mixed strategy is given as a pair $p, q \in [0,1]^2$. That is, define the expressions $\mathbb{E}u_1(p,q)$ and $\mathbb{E}u_2(p,q)$**

Let $\Sigma_1 = \{A, B\}$ and $\Sigma_2 = \{X, Y\}$. Let $\mathcal{D}_1 = p(A) + 1 - p(B)$ and $\mathcal{D}_2 = q(X) + 1 - q(Y)$ be two mixed strategies (distributions over actions) over Σ_1 and Σ_2. Then the final utility for player $i \in [2]$ is given by

$$\begin{aligned} u_i\left(\mathcal{D}_1, \mathcal{D}_2\right) = pq \cdot u_i(A,X) + (1-p)q \cdot u_i(B,X) \\ + p(1-q) \cdot u_i(A,Y) + (1-p)(1-q) \cdot u_i(B,Y) \end{aligned} \tag{3}$$

b **Generalise the expression you obtained in the first part to n player games, where each player $i \in N$ has pure strategy set Σ_i. Denote a mixed strategy profile $\boldsymbol{\mathcal{D}} = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$, where $\mathcal{D}_i \in \Delta(\Sigma_i)$ is a mixed strategy for player i. Use $\mathcal{D}_i(\sigma)$ to denote the probability of $\sigma \in \Sigma_i$ being played in the mixed strategy $\mathcal{D}_i$.**

$$\begin{aligned}
u_i(\boldsymbol{\mathcal{D}}) &= u_i(\mathcal{D}_1, \ldots, \mathcal{D}_n) && (4)\\
&= \sum_{\sigma_1,\ldots,\sigma_n\in\boldsymbol{\Sigma}} u_i(\sigma_1,\ldots,\sigma_n) \Pr_{\boldsymbol{X}\xleftarrow{\$}\boldsymbol{\mathcal{D}}}[X_1=\sigma_1,\ldots,X_n=\sigma_n] && (5)\\
&= \sum_{\sigma_1,\ldots,\sigma_n\in\boldsymbol{\Sigma}} u_i(\sigma_1,\ldots,\sigma_n) \Pr_{X_1\xleftarrow{\$}\mathcal{D}_1}[X_1=\sigma_1] \times\cdots\times \Pr_{X_1\xleftarrow{\$}\mathcal{D}_n}[X_n=\sigma_n] && (6)\\
&= \sum_{\sigma_1,\ldots,\sigma_n\in\boldsymbol{\Sigma}} u_i(\sigma_1,\ldots,\sigma_n)\mathcal{D}_i(\sigma_1)\times\cdots\times\mathcal{D}_n(\sigma_n) && (7)
\end{aligned}$$

The last step comes from independence as all players act simultaneously.

3 Question 3

For each of the following games:

a **identify any dominant strategies, dominant strategy equilibria, and pure Nash equilibria;**

Part a

		Player II	
		L	R
Player I	T	2, 1	0, 0
	B	0, 0	1, 2

There are no dominant strategies, and hence no dominant strategy equilibria. The pure Nash Equilibrium is at (T,L), (B,R).

Part b

		Player II	
		L	R
Player I	T	0, 0	3, 5
	B	4, 4	0, 3

There are no dominant strategies, and hence no dominant strategy equilibria. The pure Nash Equilibrium is at (B,L), (T,R).

b **identify outcomes that are Pareto efficient, that maximise utilitarian social welfare, and that maximise egalitarian social welfare**

Part a

		Player II	
		L	R
Player I	T	2, 1	0, 0
	B	0, 0	1, 2

(T,L) and (B,R) are both Pareto efficient, and maximise utilitarian welfare

$$\max_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \min\{u_i(\boldsymbol{\sigma}) | i \in N\} = \max_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} = \{1, 0, 0, 1\} = 1$$

Both (T,L) and (B,R) have maximise egalitarian social welfare.

Part b

		Player II	
		L	R
Player I	T	0, 0	3, 5
	B	4, 4	0, 3

(T,R) and (B, L) maximise utilitarian welfare and are pareto efficient.

$$\max_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} \min\{u_i(\boldsymbol{\sigma}) | i \in N\} = \max_{\boldsymbol{\sigma} \in \boldsymbol{\Sigma}} = \{0, 3, 4, 0\} = 4$$

(B, L) maximises egalitarian welfare.

c **Apply the principle of indifference to identify any fully mixed Nash equilibria**

Part a

		Player II	
		L	R
Player I	T	2, 1	0, 0
	B	0, 0	1, 2

$$
\begin{aligned}
\mathbb{E}[u_1(T,q)] &= \mathbb{E}[u_1(B,q)] &(8)\\
q \cdot u_1(T,L) + (1-q) \cdot u_1(T,R) &= q \cdot u_1(B,L) + (1-q) \cdot u_1(B,R) &(9)\\
2q + (1-q)0 &= q0 + (1-q) &(10)\\
q &= \frac{1}{3} &(11)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[u_2(p,L)] &= \mathbb{E}[u_1(p,R)] &(12)\\
pu_2(T,L) + (1-p)u_2(B,L) &= pu_2(T,R) + (1-p)u_2(B,R) &(13)\\
1p &= 2(1-p) &(14)\\
p &= \frac{2}{3} &(15)
\end{aligned}
$$

Part b

		Player II	
		L	R
Player I	T	0, 0	3, 5
	B	4, 4	0, 3

$$
\begin{aligned}
\mathbb{E}[u_1(T,q)] &= \mathbb{E}[u_1(B,q)] &(16)\\
q \cdot u_1(T,L) + (1-q) \cdot u_1(T,R) &= q \cdot u_1(B,L) + (1-q) \cdot u_1(B,R) &(17)\\
3(1-q) &= 4q &(18)\\
q &= \frac{3}{7} &(19)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[u_2(p,L)] &= \mathbb{E}[u_1(p,R)] &(20)\\
pu_2(T,L) + (1-p)u_2(B,L) &= pu_2(T,R) + (1-p)u_2(B,R) &(21)\\
4(1-p) &= 5p + 3(1-p) &(22)\\
p &= \frac{1}{6} &(23)
\end{aligned}
$$

d **Compute the expected utility of each player for each fully mixed strategy equilibrium you identify**

Part a

$$
\begin{aligned}
\mathbb{E}[u_1(p,q)] &= pq \cdot u_1(T,L) + (1-p)qu_1(B,L) + p(1-q)u_1(T,R) + (1-p)(1-q)u_1(B,R) &(24)\\
&= \frac{2}{3} &(25)
\end{aligned}
$$

$$\mathbb{E}[u_2(p,q)] = pq \cdot u_2(T,L) + (1-p)qu_2(B,L) + p(1-q)u_2(T,R) + (1-p)(1-q)u_2(B,R) \quad (26)$$
$$= \frac{2}{3} \quad (27)$$

Part b Doing the same thing we get $\mathbb{E}[u_1(p,q)] = \frac{12}{7}$ and $\mathbb{E}[u_2(p,q)] = \frac{10}{3}$

e **sketch the best response curves of the players in each game (no need to make it fancy – a sketch will do)**

Part A

Player II

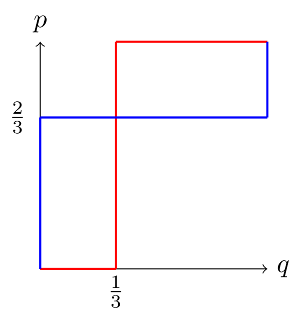
Player I		L	R
	T	2, 1	0, 0
	B	0, 0	1, 2

$\mathsf{BR}_1(q) = \arg\max_{p\in[0,1]} u_1(p,q)$

Solving the base cases, when $q = 0$ i.e. player II plays R, we have $p = 0$ as player I will pick B. When $q = 1$, $p = 1$ as player II picks L and player I will pick T to maximise its utility.

$$\mathsf{BR}_1(q) = \begin{cases} 0 & \text{if } q < \frac{1}{3} \\ [0,1] & \text{if } q = \frac{1}{3} \\ 1 & \text{if } q > \frac{1}{3} \end{cases}$$

$$\mathsf{BR}_2(p) = \begin{cases} 0 & \text{if } p < \frac{2}{3} \\ [0,1] & \text{if } p = \frac{2}{3} \\ 1 & \text{if } p > \frac{2}{3} \end{cases}$$



Part B

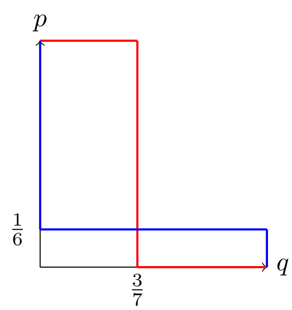
Player II

Player I		L	R
	T	0, 0	3, 5
	B	4, 4	0, 3

Solving the base cases, when $q = 0$ i.e. player II plays R, we have $p = 1$ as player I will pick T. When $q = 1$, $p = 0$ as player II picks L and player I will pick B to maximise its utility.

$$\mathsf{BR}_1(q) = \begin{cases} 1 & \text{if } q < \frac{3}{7} \\ [0,1] & \text{if } q = \frac{3}{7} \\ 0 & \text{if } q > \frac{3}{7} \end{cases}$$

$$\mathsf{BR}_2(p) = \begin{cases} 1 & \text{if } p < \frac{1}{6} \\ [0,1] & \text{if } p = \frac{1}{6} \\ 0 & \text{if } p > \frac{1}{6} \end{cases}$$



4 Question 4

We will prove this statement for the general game consisting of $N \geq 2$ players. For $N = 2$, the notation below would simplify to $\boldsymbol{\mathcal{D}}^* = (\mathcal{D}_1^*, \mathcal{D}_2^*)$, where $\mathcal{D}_1^* = p(T) + 1 - p(B)$ and $\mathcal{D}_2^* = q(L) + 1 - p(R)$.

Intuition: Imagine this was not the case, then and the utility of a_i was greater than b_i. Then, by the monotonicity axiom, I am always incentivised to play a_i with greater probability. So I'll want to maximise the probability of playing a_i and set it to 1 and the probability of b_i to 0.

Proving the $\Rightarrow$ direction.
We assume that $\boldsymbol{\mathcal{D}}^*$ is a Nash equilibrium. Fix player $i \in N$, such that there are two actions $a_i, b_i \in \Sigma_i$ such that[1] $\mathcal{D}_i^*(a_i), \mathcal{D}_i^*(b_i) > 0$. We want to show that

$$\mathbb{E}[u_i(a_i, \boldsymbol{\mathcal{D}}_{-i}^*)] = \mathbb{E}[u_i(b_i, \boldsymbol{\mathcal{D}}_{-i}^*)]$$

We do this by proving the contrapositive! Assume that for all $i \in N$

$$\mathbb{E}[u_i(a_i, \boldsymbol{\mathcal{D}}_{-i}^*)] > \mathbb{E}[u_i(b_i, \boldsymbol{\mathcal{D}}_{-i}^*)] \tag{28}$$

Then we want to show that $\boldsymbol{\mathcal{D}}^*$ is not a Nash equilibrium i.e. that $\mathcal{D}_i^*$ is not the best response to $\boldsymbol{\mathcal{D}}_{-i}^*$.

Define a mixed action σ_i in the following way, for any $t_i \in \Sigma_i$

$$\mathcal{D}_i(t_i) = \begin{cases} \mathcal{D}_i^*(t_i) & \text{if } t_i \neq a \text{ and } t_i \neq b \\ 0, & \text{if } t_i = b_i \\ \mathcal{D}_i^*(a_i) + \mathcal{D}_i^*(b_i), & \text{if } t_i = a_i \end{cases}$$

Now,

[1] In the specific problem, we are given that $p, q \in (0,1)$ and thus both actions have a chance of being played.

$$\begin{aligned}
u_i(\mathcal{D}_i, \boldsymbol{\mathcal{D}}^*_{-i}) &= \sum_{t_i \in \Sigma_i} \mathcal{D}_i(t_i) \cdot \mathbb{E}\left[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] && (29)\\
&= \Big(\mathcal{D}^*_i(a_i) + \mathcal{D}^*_i(b_i)\Big)\mathbb{E}\left[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] + \sum_{t_i \notin \{a_i, b_i\}} \mathcal{D}_i(t_i) \cdot \mathbb{E}\left[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] && (30)\\
&= \mathcal{D}^*_i(a_i) \cdot \mathbb{E}\left[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] + \mathcal{D}^*_i(b_i) \cdot \mathbb{E}\left[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] + \sum_{t_i \notin \{a_i, b_i\}} \mathcal{D}_i(t_i) \cdot \mathbb{E}\left[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] && (31)\\
&> \mathcal{D}^*_i(a_i) \cdot \mathbb{E}\left[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] + \mathcal{D}^*_i(b_i) \cdot \mathbb{E}\left[u_i(b_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] + \sum_{t_i \notin \{a_i, b_i\}} \mathcal{D}_i(t_i) \cdot \mathbb{E}\left[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] && (32)\\
&= \sum_{t \in \Sigma_i} \mathcal{D}^*_i(t_i)\mathbb{E}\left[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})\right] && (33)\\
&= u_i(\mathcal{D}^*_i, \boldsymbol{\mathcal{D}}^*_{-i}) && (34)
\end{aligned}$$

Equation (32) comes from the assumption (28). Thus $u_i(\mathcal{D}_i, \boldsymbol{\mathcal{D}}^*_{-i}) > u_i(\mathcal{D}^*_i, \boldsymbol{\mathcal{D}}^*_{-i})$, making $\mathcal{D}_i$ a better response. Thus, making $\boldsymbol{\mathcal{D}}^*$ not a nash equilibrium.

Proving the $\Leftarrow$ direction.

For this direction we will assume[2] that a_i, b_i are the only available actions for player i i.e $\Sigma_i = \{a_i, b_i\}$, which is what the question states anyway.

We assume

$$\mathbb{E}[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})] = \mathbb{E}[u_i(b_i, \boldsymbol{\mathcal{D}}^*_{-i})] \tag{35}$$

and want to show that $\boldsymbol{\mathcal{D}}^*$ is a Nash equilibrium. This is the same as showing for all strategies $\mathcal{D}_i \in \Delta(\Sigma_i)$, we will have

$$\mathbb{E}[u_i(\mathcal{D}_i, \boldsymbol{\mathcal{D}}^*_{-i})] \leq \mathbb{E}[u_i(\mathcal{D}^*_i, \boldsymbol{\mathcal{D}}^*_{-i})]$$

Pick an arbitrary $\mathcal{D}_i$,

$$\begin{aligned}
\mathbb{E}[u_i(\mathcal{D}^*_i, \boldsymbol{\mathcal{D}}^*_{-i})] &= \sum_{t_i \in \{a_i, b_i\}} \mathcal{D}^*_i(t_i) \cdot \mathbb{E}[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})] && (36)\\
&= \mathbb{E}[u_i(a_i, \boldsymbol{\mathcal{D}}^*_{-i})] && (37)\\
&= \sum_{t_i \in \{a_i, b_i\}} \mathcal{D}_i(t_i) \cdot \mathbb{E}[u_i(t_i, \boldsymbol{\mathcal{D}}^*_{-i})] && (38)
\end{aligned}$$

Equation (36) comes from assumption (35). Thus, we show that there is no other distribution $\mathcal{D}_i$ that player i can employ that is a better response to $\boldsymbol{\mathcal{D}}^*_{-i}$ in expectation. Thus, as i is general, $(\mathcal{D}^*_i, \boldsymbol{\mathcal{D}}^*_{-i})$ must be a Nash equilibrium.

[2] The other direction does not need such an assumption

5 Question 5

Two firms, X and Y, provide a service based on machine learning. The market share of each firm is directly proportional to the quality of its service, which, in turn, is directly proportional to the size of the data set it has: thus, if X has x data points and Y has y data points, the profit of X (in £) is $\frac{x}{x+y} \cdot M$ and the profit of Y (in £) is $\frac{y}{x+y} \cdot M$, where M is the overall value of the market. Initially, firm X possesses 1 million data points and firm Y possesses 2 million data points. They are both presented with an opportunity to buy a new corpus of data, consisting of n million data points,where $n > 0$, at price P (in £), $P > 0$. If both firms express the desire to buy, each of them gets one half of the new data, i.e., $\frac{n}{2}$ points, and pays $\frac{P}{2}$.

a **Describe this setting as a two-player game, where each player's choice of actions is Buy (buy) and NotBuy (do not buy). For simplicity, assume M = 1 in this and subsequent parts.**

X \ Y	B	N
B	$\frac{n/2+1}{n+3} - \frac{P}{2}, \frac{n/2+2}{n+3} - \frac{P}{2}$	$\frac{1+n}{n+3} - P, \frac{2}{n+3}$
N	$\frac{1}{n+3}, \frac{n+2}{n+3} - P$	$\frac{1}{3}, \frac{2}{3}$

b **Suppose that n is fixed. Under what conditions on the price P is Buy a weakly dominant strategy for firm X? Under what conditions on P is Buy a weakly dominant strategy for firm Y? NB: note that we are after a weakly dominant strategy here, so must always be at least as good and in one case strictly better.**

Assume $n > 0$.

When is P a weakly dominant strategy for X? We need a case when B is strictly better than N, and a case where B is as good as N or better.

$$u_1(B,B) > u_1(N,B) \implies P < \frac{n}{n+3} \tag{39}$$
$$u_1(B,N) \geq u_1(N,N) \implies P \leq \frac{2n}{3n+9} \tag{40}$$

Simplifying (39) gives us $P < \frac{n}{n+3}$ and, simplifying (40) gives us $P \leq \frac{2n}{3n+9}$. Similarly,

$$u_2(B,B) > u_2(B,N) \implies P < \frac{n}{n+3} \tag{41}$$
$$u_2(N,B) \geq u_2(N,N) \implies P \leq \frac{n}{3(n+3)} \tag{42}$$

c **Suppose that n is fixed. Characterise the range of values of P such that (Buy , Buy) is a Nash equilibrium.**

For (B,B) to be a nash equilibrium, it needs to be the best response strategy for both players, fixing the other players actions i.e no player will want to switch. Formally, we need

$$u_1(N,B) \leq u_1(B,B) \implies \frac{1}{n+3} \leq \frac{n/2+1}{n+3} - \frac{P}{2} \tag{43}$$
$$u_2(B,N) \leq u_2(B,B) \implies \frac{2}{n+3} \leq \frac{n/2+2}{n+3} - \frac{P}{2} \tag{44}$$

Solving both we get $p \leq \frac{n}{n+3}$.

d **Suppose that n is fixed. Characterise the range of values of P such that (NotBuy , NotBuy) is a Nash equilibrium**

We need

$$u_1(B,N) \leq u_1(N,N) \implies p \geq \frac{1+n}{3+n} - \frac{1}{3} \tag{45}$$

$$u_2(N,B) \leq u_2(N,N) \implies p > \frac{2+n}{3+n} - \frac{2}{3} \tag{46}$$

If (45) is satisfied we get (46) as well. So we need $p \geq \frac{1+n}{3+n} - \frac{1}{3}$ which also guarantees $p > \frac{2+n}{3+n} - \frac{2}{3}$

e **Suppose that n is fixed. Are there values of P such that (Buy , NotBuy) or (NotBuy , Buy) is a Nash equilibrium?**

No!

$$u_1(N,N) \leq u_1(B,N) \implies P \leq \frac{n}{3n+9} \tag{47}$$

$$u_2(B,B) \leq u_2(B,N) \implies P \geq \frac{3n}{3n+9} \tag{48}$$

And both those contradict each other. A similar argument holds for (N, B). See below links to Mathematica that were used to reduce these equations.

(47): https://www.wolframalpha.com/input?i=solve+for+p%3A+1%2F3+%3C%3D+%28n%2F2%2B1%29%2F%28n%2B3%29+-+p%2F2

(48): https://www.wolframalpha.com/input?i=solve+for+p%3A+%28n%2F2%2B2%29%2F%28n%2B3%29+-+p%2F2+%3C%3D+2%2F%28n%2B3%29

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