Preferences, Utilities, and Decisions

Tutorial 1 Solutions

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Some Guidelines On Notation We use Ω to denote the set of outcomes and $\Delta(\Omega)$ to denote probability distributions over outcomes. In the lecture slides $\Delta(\Omega)$ is described as the set of all lotteries. Thus $\mathcal{D} \in \Delta(\Omega)$ describes a lottery (or probability distribution) over Ω. When denote as $x \xleftarrow{\$} \mathcal{D}$ as x was sampled according to the probability distribution specified by $\mathcal{D}$.

1 Solutions

Question 1 Recall the example of lexicographic preferences from the lecture. The attributes are colour, engine type, and nationality, with the ranking

$$\text{colour} > \text{engine} > \text{nationality}$$

The ordering for each of attribute is:

$$\text{red} \succ \text{blue} \succ \text{green}$$
$$\text{electric} \succ \text{petrol} \succ \text{diesel}$$
$$\text{German} \succ \text{French} \succ \text{UK}$$

Define a utility function that takes three inputs (colour, engine type, and nationality) and gives as output a real number, such that this utility function corresponds to the preference ordering defined above. Don't do this by enumerating all 27 cases! Argue for the correctness of your function.

Solution 1 Assign a utility score lexicographically for each attribute, such as red=9, blue=8, green=7, electric=6, petrol=5, diesel=4, german=3, french=2, uk=1.

Let $a \in \{\text{red}, \text{blue}, \text{green}\}$, $b \in \{\text{electric}, \text{petrol}, \text{diesel}\}$ and $c \in \{\text{german}, \text{french}, \text{uk}\}$then define,

$$u(a, b, c) = a^3 + b^2 + c$$

There are many possible utility functions that work. The key thing you have to keep track of is that contribution of (blue, electric, german) should not override (red, diesel, uk). Similarly, A german petrol score should override a UK car with electric engine. And so on.

Write a little computer program in your preferred language to verify that your solution is accurate.

Listing 1: Output for different evaluations of u

```
red, electric, german       768
red, electric, french       767
red, electric, british      766
red, petrol, german         757
red, petrol, french         756
red, petrol, british        755
red, diesel, german         748
red, diesel, french         747
red, diesel, british        746
blue, electric, german      551
blue, electric, french      550
blue, electric, british     549
blue, petrol, german        540
blue, petrol, french        539
blue, petrol, british       538
blue, diesel, german        531
blue, diesel, french        530
blue, diesel, british       529
green, electric, german     382
green, electric, french     381
green, electric, british    380
green, petrol, german       371
green, petrol, french       370
green, petrol, british      369
green, diesel, german       362
green, diesel, french       361
green, diesel, british      360
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Question 2 To make things simple in the lectures, we assumed our set of outcomes Ω was finite. Let's now look at an example where this is not the case. Let the set of outcomes $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers.

Now ,define a preference relation $\succeq \subseteq \Omega \times \Omega$ by:

$$(x_1, x_2) \succeq (y_1, y_2) \iff x_1 > y_1 \text{ or } x_1 = y_1 \wedge x_2 \geq y_2$$

(a) Prove that the relation $\succeq$ defined in this way is indeed a properly defined preference relation.

(b) Prove that there can exist no utility function $u : \Omega \to \mathbb{R}$ representing $\succeq$.

Solution 2 **Part a**: To show that $\succeq$ is a preference relation we must show that it is (1) complete (2) transitive (3) Reflexive

Completeness:Fix any $a = (x_1, x_2)$ and $b = (y_1, y_2)$ such that $a, b \in \Omega$. If $x_1 \neq y_1$, then as $x_1 \in \mathbb{R}$, we must have either $x_1 < y_1$ or $x_1 > y_1$. This would lead us to conclude $b \succeq a$ or $a \succeq b$, respectively. If $x_1 = y_1$, then the same logic applies to x_2 and y_2 as both $x_2, y_2 \in \mathbb{R}$. As a and b were arbitrary, we have that $\succeq$ is complete.

Transitivity:Fix $a = (x_1, x_2)$, $b = (y_1, y_2)$ and $c = (z_1, z_2)$ where $a, b, c \in \Omega$. Assume that $a \succeq b$ and $b \succeq c$. Then we must show that $a \succeq c$.

As $a \succeq b$, we have one of the two possibilities.

$$x_1 > y_1 \tag{1}$$
$$x_1 = y_1 \wedge x_2 \geq y_2 \tag{2}$$

Similarly, as $b \succeq c$, we have one of

$$y_1 > z_1 \tag{3}$$
$$y_1 = z_1 \wedge y_2 \geq y_2 \tag{4}$$

This gives us 4 possible outcomes, which we list below:

Assume that 1 and 3 hold. In this case by transitivity of $\mathbb{R}$, we have $x_1 > z_1$, and therefore $a \succeq c$. Assume that 2 and 3 hold Then we have $x_1 = y_1 > z_1$, and thus once again we have $a \succeq c$. Assume that 1 and 4 hold. We have $x_1 > y_1 = z_1$ and thus $a \succeq c$. Finally, assume that 2 and 4 holds. Then we have $x_1 = y_1 = z_1$ and $x_2 \geq y_2 \geq z_2$, and thus $a \succeq c$.

Reflectivity Fix any $x_1, x_2 \in \Omega$, then as $x_1 = x_1 \wedge x_2 \geq x_2$, we have $(x_1, x_2) \succeq (x_1, x_2)$.

Part b: The proof to this statement relies on two mathematical facts about the real number line and the set of rationals ($\mathbb{Q}$). The set $\mathbb{R}_+$ is un countable and the set $\mathbb{Q}$ is countable. We will show that if there is an utility function for the relation $\succeq$, then the set $\mathbb{R}_+$ MUST be countable. Thus we get what we want by contradiction.

Fix an arbitrary utility function $u : \Omega \to \mathbb{R}_+$. For any $x \in \mathbb{R}_+$, define $a(x) = u(x, 0)$ and $b(x) = u(x, 1)$. As u is a utility function for $\succeq$, we have $a(x) < b(x)$. As both $a(x), b(x) \in \mathbb{R}_+$, we can always find a rational number r such that $a(x) < r < b(x)$. Thus for each x, define a function $f : \mathbb{R}_+ \to \mathbb{Q}$, such that $f(x) = r$. Now if we show that f is bijective, then it means for every $x \in \mathbb{R}_+$ there is a unique $r \in \mathbb{Q}$, this would imply that $\mathbb{R}_+$ is countable as $\mathbb{Q}$ is countable. In order to show f is bijective, pick $\tilde{x} \neq x$ and assume without loss of generality that $\tilde{x} > x$. Then as u is a utility function, we have $a(x) < r(x) < b(x) < a(\tilde{x}) < b(\tilde{x})$. Once again we can find a $\tilde{r} \in \mathbb{Q}$ such that $a(\tilde{x}) < \tilde{r} < b(\tilde{x})$. Thus, we can infer that $r \neq \tilde{r}$, which proves that f maps each unique $x \in \mathbb{R}_+$ to a unique rational number.

$\square$.

Question 3 You have three entertainment options: football, pub, or rowing. The utility you obtain from these will depend on whether it rains or not. The utilities you get from these outcomes are as follows:

Activity	Utility if Rain	Utility if no rain
rowing	0	1
football	1	2
pub	3	0

You have a decision task ahead: to choose which activity to undertake. In what follows, let p_{rain} denote the probability of rain.

(a) Formulate this as a problem of decision making under uncertainty.

(b) Can you identify an alternative that you would never choose, irrespective of the value of p_{rain}.

(c) Can you write down of a rule that shows what the best thing to do is, as a function of p_{rain}?

Solution 3 part a: We have a set of outcomes $\Omega = \{\text{Rain}, \text{No Rain}\}$ and a set of actions $\Sigma = \{\text{Football}, \text{Pub}, \text{Rowing}\}$. Then we have a lottery l where

$$l = p_{\mathsf{rain}} \cdot \text{Rain} + (1 - p_{\mathsf{rain}}) \cdot \text{NoRain}$$

The utility based on the outcome is given by the table in the question.

part b: I would never choose rowing. To see why, we write down the expected utility for each activity. Let $p = p_{\mathsf{rain}}$.

$$\mathbb{E}_p\text{rowing} = 1 - p \quad (5)$$
$$\mathbb{E}_p\text{football} = 2 - p \quad (6)$$
$$\mathbb{E}_p\text{pub} = 3p \quad (7)$$

Equation (6) is strictly greater than (5) for all $p \in [0, 1]$.

part c: The following rule identifies my strategies based on p_{rain}

$$\text{Activity} = \begin{cases} \text{pub} & \text{if } p_{\mathsf{rain}} > \frac{1}{2} \\ \text{football} & \text{if } p_{\mathsf{rain}} < \frac{1}{2} \\ \text{indifferent between pub and football} & \text{if } p_{\mathsf{rain}} = \frac{1}{2} \end{cases}$$

Question 4 Suppose $\Omega = \{A, B\}$ and $A \succ B$. Then I claim that, for all values $p \in (0, 1]$, we have

$$pA + (1 - p)B \succ B$$

Argue that if $\succ$ satisfies the Von Neumann and Morgenstern axioms, then this property will indeed hold. You should make clear in your answer which of the Von Neumann and Morgenstern axioms you are appealing to.

Solution 4 We use the monotonicity property which states that if you prefer A over B, then you will prefer to maximise the probability of getting A over B.

Define lotteries $\mathcal{D}_1 = \{p(A), 1 - p(B)\}$ and $\mathcal{D}_2 = \{0(A), 1(B)\}$ where p is the probability of outcome A occurring according to lottery $\mathcal{D}_1$ and 0 is the probability of outcome A under $\mathcal{D}_2$. As $p > 0$ (by monotonicity), we can conclude $pA + (1 - p)B \succ B$. □

Question 5 Consider the following four lotteries with monetary rewards:

$$\mathcal{D}_1 = \frac{1}{2}\$100 + \frac{1}{2}\$0 \quad (8)$$
$$\mathcal{D}_2 = \$50 \quad (9)$$
$$\mathcal{D}_3 = \frac{1}{20}\$100 + \frac{19}{20}\$0 \quad (10)$$
$$\mathcal{D}_4 = \frac{1}{10}\$50 + \frac{9}{10}\$0 \quad (11)$$

Now suppose I claim that my preferences satisfy both $\mathcal{D}_2 \succ \mathcal{D}_1$ and $\mathcal{D}_3 \succ \mathcal{D}_4$. Show that my preferences in this case do not satisfy the Von Neumann and Morgenstern axioms.

Solution 5 Let $x = u(\$100), y = u(\$0) and z = u(\$50)$ Assume the preferences satisfy the axioms, then $\mathcal{D}_2 \succ \mathcal{D}_1$ implies

$$\frac{1}{2}x + \frac{1}{2}y \leq z \tag{12}$$
$$\frac{1}{20}x + \frac{1}{20}y < \frac{1}{10}z \tag{13}$$

Now if $\mathcal{D}_3 \succ \mathcal{D}_4$, then we have

$$\frac{1}{20}x + \frac{19}{20}y > \frac{1}{10}z + \frac{9}{10}y \tag{14}$$
$$\frac{1}{20}x + \frac{1}{20}y > \frac{1}{10}z \tag{15}$$

Contradiction! $\square$

Question 6 Suppose $\Omega = \{A, B, C, D\}$, and consider the following four lotteries over Ω.

$$\mathcal{D}_1 = \frac{3}{5}\mathrm{A} + \frac{2}{5}\mathrm{D} \tag{16}$$
$$\mathcal{D}_2 = \frac{3}{4}\mathrm{A} + \frac{1}{4}\mathrm{C} \tag{17}$$
$$\mathcal{D}_3 = \frac{2}{5}\mathrm{A} + \frac{1}{5}\mathrm{B} + \frac{1}{5}\mathrm{C} + \frac{1}{5}\mathrm{D} \tag{18}$$
$$\mathcal{D}_4 = \frac{2}{5}\mathrm{A} + \frac{3}{5}\mathrm{C} \tag{19}$$

Suppose that a Von Neumann and Morgenstern preference relation $\succ \subseteq \Delta(\Omega) \times \Delta(\Omega)$ satisfies the following properties:

$$\mathrm{C} \sim \mathcal{D}_1 \tag{20}$$
$$\mathrm{B} \sim \mathcal{D}_2 \tag{21}$$
$$\mathrm{A} \succ \mathrm{D} \tag{22}$$

How are $\mathcal{D}_3$ and $\mathcal{D}_4$ ranked?

Solution 6 If A $\succ$ B, we can infer that $\mathcal{D}_4 \succ \mathcal{D}_3$. To see why, observe $\mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_3} u(\omega) = \frac{2}{5}u(A) + \frac{1}{5}u(B) + \frac{1}{5}u(C) + \frac{1}{5}u(D)$ and $\mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_4} u(\omega) = \frac{2}{5}u(A) + \frac{3}{5}u(C)$. Thus for $\mathcal{D}_4 \succ \mathcal{D}_3$, we need $\mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_4} u(\omega) - \mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_r} u(\omega) > 0$.

$$\begin{aligned}\mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_3} u(\omega) - \mathbb{E}_{\omega \xleftarrow{\$} \mathcal{D}_4} u(\omega) &= \frac{1}{5}(u(B) - 2u(C) + u(D)) \\ &= -3/50\Big(u(A) - u(D)\Big)\end{aligned} \tag{23}$$

Equation (23) comes from substituting[1] $u(c) = 3/5u(a) + 2/5u(d)$ and $u(b) = 3/4u(a) + 1/4u(c)$ which come from equations (20) and (21). Finally as we have from (22) that $u(A) - u(D) > 0$ we get what we want. □

Question 7 Suppose a person whose preferences satisfy the Von Neumann and Morgenstern axioms says that with respect to lotteries $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$, her preferences are

$$\mathcal{D}_1 \succeq \mathcal{D}_2 \text{ and } \mathcal{D}_3 \succeq \mathcal{D}_4$$

Consider the following property. For all $p \in [0, 1]$ we have

$$p\mathcal{D}_1 + (1-p)\mathcal{D}_3 \succeq p\mathcal{D}_2 + (1-p)\mathcal{D}_4$$

Do you think this property should hold? If so, can you provide an argument that it does with respect to the von Neumann and Morgenstern axioms?

Solution 7 The condition holds by independence We have

$$\mathcal{D}_1 \succeq \mathcal{D}_2 \implies p\mathcal{D}_1 + (1-p)\mathcal{D}_3 \succeq p\mathcal{D}_2 + (1-p)\mathcal{D}_3 \tag{24}$$

(24) comes from Independence or substitution (introducing $\mathcal{D}_3$ does not change my preferences over $\mathcal{D}_1$ and $\mathcal{D}_2$). As we have $\mathcal{D}_3 \succeq \mathcal{D}_4$ we must have

$$p\mathcal{D}_1 + (1-p)\mathcal{D}_3 \succeq p\mathcal{D}_2 + (1-p)\mathcal{D}_3 \implies p\mathcal{D}_1 + (1-p)\mathcal{D}_3 \succeq p\mathcal{D}_2 + (1-p)\mathcal{D}_4$$

The last inequality comes from the fact that $u(\mathcal{D}_3) > u(\mathcal{D}_4)$ and as p and $(1-p)$ are fixed for both. □

Question 8 Consider the following scenario: You toss a fair coin repeatedly, until the coin shows heads for the first time. You are then paid £2^k, where k is the number of times the coin was tossed.

(a) Express this as a lottery (in which the set of outcomes is infinite).

(b) Assuming utility is expected monetary reward, compute the expected utility of this lottery.

(c) Now assume that the utility function u over monetary outcomes is such that $u(n) = \log_2 n$, where n is the amount of money earned. Show that the agent's expected utility of this game is upward bounded.

(d) By giving examples, argue that the agent with the utility function as in the previous part (i.e., logarithmic) is risk-averse.

Solution 8 **Part a**: The set of outcomes is $\Omega = \{2^1, 2^2, 2^3, \ldots, \}$ and $|\Omega|$ is infinite. The lottery over outcomes follows a geometric distribution[2] with parameter $p = 1/2$ which is the probability of seeing heads.

$$\Pr_{\omega \xleftarrow{\$} \mathsf{Geometric}(p)} \Big[\omega = k\Big] = (1-p)^{k-1} p$$

[1] Plug the constaints into wolfram alpha or mathematica and you get what you need. See https://www.wolframalpha.com/input?i=simplify+%28b%2F5+%2B+c%2F5+%2Bd%2F5%29+-+3%2F5c+where+c+%3D+3%2F5a+%2B+2%2F5d+and+b+%3D+3%2F4a+%2B+1%2F4c

[2] https://www.wikiwand.com/en/Geometric_distribution

Part b: Let $n = 2^k$ where $k \xleftarrow{\$} \mathsf{Geometric}(p)$.

$$\mathbb{E}_{x \xleftarrow{\$} \mathsf{Geometric}(p)} n = \sum_{k=1} (1-p)^{k-1} p 2^k \tag{25}$$
$$= 2p \sum_{k=1} (2-2p)^{k-1} \tag{26}$$
$$= \frac{2p}{2p-1} \tag{27}$$
$$= \infty \tag{28}$$

(28) comes by plugging in $p = 1/2$ and seeing that it is unbounded.

Part c and d:

Let $p(i)$ denote shorthand for $\Pr_{x \xleftarrow{\$} \mathsf{Geometric}(p)}[x = i]$. It is easy to see that

$$\mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u(x) = \sum_{i=1}^{\infty} p(i) \cdot \log_2 2^i \tag{29}$$
$$= \sum_{i=1}^{\infty} p(i) i \tag{30}$$
$$= \frac{1}{p} \tag{31}$$
$$< \infty \tag{32}$$

Equation (31) comes from the mean of a geometric random variable (see `https://online.stat.psu.edu/stat414/lesson/11/11.2` for a full derivation).

Alternatively, one can prove that the agent is risk averse by showing that the utility function is strictly concave as a function of n, which $\log_2(n)$ is. By Jensens inequality it is easy to show that this statement is the same as showing the above. In fact there is a theorem that states that a utility function to be risk averse if and only if it is a strictly concave function. (See supplementary notes) □

Question 9 The following story, albeit slightly morbid, is nevertheless apparently true. In the second world war, a US bomber squadron was based 3000km from their target. The target was so far away that fighter plane escorts were impossible, making missions even more than usually dangerous. Planes could only carry a few bombs on each mission, so that they could carry enough fuel to return to base. Pilots were scheduled to fly 30 missions before returning to the USA, but on average only half of the pilots survived all 30 missions. Logistics experts came up with the following proposal. Each plane would carry a much heavier bomb load – but only enough fuel to fly one way. Thus, each mission would be a suicide mission. However, the increased bomb load would mean that far fewer missions would be needed, allowing 75% of the pilots to return home. The other 25% of pilots, who had to fly the missions, would face certain death. Those to fly the missions would be selected randomly. Every pilot who was presented with the new proposal rejected it in favour of the status quo.

(a) Formulate the above two scenarios as lotteries within the Von Neumann and Morgenstern framework, in which there are just two outcomes, live and die, and such that live $\succ$ die. Show that, with preferences as expressed in the scenario above, the pilots violate Von Neumann and Morgenstern's axioms. Which of the axioms did the pilots violate when they made their choice?

(b) Now assume there are three outcomes, with associated preferences as follows:

$$\texttt{live with honour} \succ \texttt{die with honorlive with no honour}$$

So: living with honour would mean flying a mission and surviving; to die with honour would mean flying a mission and being killed; and living with- out honour would mean living because somebody else had flown a mission to certain death. Now reformulate the above scenarios as lotteries using these preferences. Do the pilots violate the Von Neumann and Morgenstern axioms?

(c) Are the preferences of airmen in this example lexicographic? If so can you give relevant attributes and explain how they are ranked?

(d) What factors do you think may have influenced the pilot's decisions?

Solution 9 **Part a**: Define outcomes $\Omega = \{\mathsf{live}, \mathsf{die}\}$, and the original distribution $\mathcal{D}_1 = \frac{1}{2}\mathsf{live} + \frac{1}{2}\mathsf{die}$. Under the new rule we now have $\mathcal{D}_2 = \frac{3}{4}\mathsf{live} + \frac{1}{4}\mathsf{die}$. If we assume $\mathsf{live} \succ \mathsf{die}$, then by monotonicity we should have $\mathcal{D}_2 \succ \mathcal{D}_1$ as $\frac{3}{4} > \frac{1}{2}$. But this is not the case so, the preferences do not satisfy the Von Neumann and Morgenstern's axioms.

Part b:

$\Omega = \{\texttt{live with honour}, \texttt{die with honor}, \texttt{live with no honour}\}$

This preference model satisfies the Von Neumann and Morgenstern's axioms. As the soldiers prefer to die with honour over living without honour, they would never accept a proposal to send more of their comrades to war while avoiding dying themselves. So the new regime lottery would imply

$$\mathcal{D}_3 = \frac{1}{4} \cdot \texttt{die with honor} + 0 \cdot \texttt{live with honour} + \frac{3}{4} \cdot \texttt{live with no honour}$$

while the old regime is

$$\mathcal{D}_4 = \frac{1}{2} \cdot \texttt{die with honor} + \frac{1}{2} \cdot \texttt{live with honour} + 0 \cdot \texttt{live with no honour}$$

As $\frac{1}{2} > 0$ and $\frac{1}{2} > \frac{1}{4}$, by the monotonicity property again, we see that it makes sense that $\mathcal{D}_4 \succ \mathcal{D}_3$, and it satisfies the axioms.

Part c: Preferences are lexicographic if outcomes can be characterised by an ordered set of attributes, where each attribute has its own ordering. The attributes are $\{\text{pride}, \text{mortality}\}$ and the ordering is pride $\succ$ mortality. In each attribute, we have

1. **pride**: honour $\succ$ no honour.
2. **mortality**: life $\succ$ death.

So, the soldiers would prefer to live with honour above all, regardless of whether they live or die. If they are guaranteed honour, they would rather live over die.

Part d: The surviving soldiers might have to go to war in the future and they do not want their compatriots to know that they refused to go, as it might erode trust. □