Utility Theory

Notes For Tutorial 1

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Abstract

Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications. They may contain errors and will almost certainly contain typos.

These notes formally introduce the idea of preferences and what conditions preferences must satisfy so that they can be represented by a function that map outcomes to the real number line. Notes mostly based on [MZS20, Chapter 2] and a few other proofs from relevant text.

1 Preferences And Utility

A game is a mathematical model of a situation of interactive decision making, in which every player wishes to attain the "best possible outcome", knowing that the other player wants to do the same thing. The idea of utility theory is to formalise what it means to have the "best possible outcome". For any game let Ω denote the set of possible outcomes. Apriori, we make no assumptions about Ω, it may be finite, infinite or even uncountable. Once we have a set of outcomes defined, we need to way to describe how the outcomes compare with each other for any given player. This is established by preference relations.

Definition 1 (Preference Relation) A preference relation over a set of outcomes Ω is a binary relation denoted by $\succeq$. Formally, a binary relation is a subset of $\succeq \subseteq \Omega \times \Omega$, but instead of writing $(x, y) \in \succeq$, we write $x \succeq y$ and read it as the player prefers x over y. We say a player strictly prefers x over y if

$$x \succ y \iff x \succeq y \land y \not\succeq x$$

We say a player is indifferent between x and y (denoted by $x \sim y$) if and only if

$$x \succeq y \land y \succeq x$$

Axioms For A Preference Relation

Axiom 1 (Complete) A preference relation $\succeq$ is complete if for any pair of outcomes $x, y \in \Omega$, we have either $x \succeq y$ or $y \succeq x$, or both.

Axiom 2 (Reflexive) A preference relation $\succeq$ is reflexive if for any $x \in \Omega$, we have either $x \succeq x$.

Axiom 3 (Transitive) A preference relation $\succeq$ is transitive if $A \succeq B$ and $B \succeq C$, we have $A \succeq C$.

At this point we should be able to fully compare outcomes with each other, and determine what the best possible outcome is. However, preference relations as described are a combinatorial object. Since, we have well-established tools (such as calculus) for using real numbers, we would like is to instead, have a way of mapping preferences over outcomes to the real number line. This is what a utility function does.

Definition 2 (Utility Function) Let Ω be a set of outcomes and $\succeq$ be a complete, transitive and reflexive preference relation over Ω. A function $u : \Omega \to \mathbb{R}$ is a utility function representing $\succeq$, if for all $x, y \in \Omega$

$$x \succeq y \iff u(x) \geq u(y) \tag{1}$$

Theorem 1 If u is a utility function representing $\succeq$, and $v : \mathbb{R} \to \mathbb{R}$ is a strictly monotone function, then $v \circ u$ is also a utility function for $\succeq$.

Proof. Proving the "If" direction. Assume $x \succeq y$, then $u(x) > u(y)$, and by monotonicity, $v(u(x)) > v(u(y))$. Proving the "Only If" direction Assume $v(u(x)) > v(u(y))$, then once again by the monotone the images are unique, and so $u(x) > u(y)$, implying $x \succeq y$. □

Why do we care about this theorem? Often we will take a utility function, and consider the log of the utility function instead. Helps converting products into sums which are easier to analyse.

Question 1 When can be sure that a preference relation $\succeq$ can be represented by a utility function? Problem 2 on the Tutorial 1 (Lexicographic preferences) showed that the preference function could not have utility function despite satisfying all the axioms. Can we come up with a set of rules?

Theorem 2 (Finite set of outcomes Ω, generalisation of Slide 53 of lecture notes) Suppose X is finite, then $\succeq$ is a preference relation if and only if there exists some utility function $u : \Omega \to \mathbb{R}$ that represents $\succeq$.

Proof. I'll only sketch the proof but the complete proof is constructive and can be found [Rig18, Page 5].

Define $\preceq (x) = \{y \in \Omega : x \succeq y\}$ as lower contour set of x and $\succeq (x) = \{y \in \Omega : y \succeq x\}$ be the upper contour set of x. Setting $u(x) = | \preceq (x)|$ suffices to show that it is a valid utility function and then use the fact the contour sets must also be finite, as Ω is finite. □

We do not actually need the set to be finite. Even outcomes that are countable have utility functions.

Theorem 3 (Countable set of outcomes Ω) Suppose X is countable, then $\succeq$ is a preference relation if and only if there exists some utility function $u : \Omega \to \mathbb{R}$ that represents $\succeq$.

Proof. See [Rub12, Page 14]. Use induction and transitive property. □

Question 2 But what about uncountable outcomes? Are we doomed?

It turns out that we had another property of continuity, then even preferences over uncountable outcomes have utility representations.

Definition 3 (continuous preferences) A preference relation $\succeq$ is continuous if the sets $\preceq (x) = \{y \in \Omega : x \succeq y\}$ and $\succeq (x) = \{y \in \Omega : y \succeq x\}$ are closed for all $x \in \Omega$.

With continuity, we get one of the most famous results in utility theory.

Theorem 4 (Debreuis Representation Theorem) Suppose $\Omega \subseteq \mathbb{R}^n$. The binary relation $\succeq$ on Ω is complete, transitive, and continuous *if and only if* there exists a continuous utility representation $u : \Omega \to \mathbb{R}$.

Proof. See [Rig18, Slide 12]. □

2 Lotteries

Much of life is not deterministic. When the outcome is not deterministic we say we have a lottery. Lotteries are just a fancy word for probability distributions. A simple lottery $\mathcal{D}$ is a probability distribution over the set of outcomes Ω. We denote with $\Delta(\Omega)$ the set of all such probability distributions. A compound lottery, is simply a probability distribution $\widetilde{\mathcal{D}}$ over a set of probability distributions i.e. $\widetilde{\mathcal{D}} \in \Delta\Big(\Delta(\Omega)\Big)$. As convention, and unless otherwise stated, we will denote simple lotteries with $\mathcal{D}$ and compound lotteries with $\widetilde{\mathcal{D}}$. Additionally, we use the following notation to express that event A occurs with probability p_A and event B occurs with probability p_B when sampling from a simple lottery $\mathcal{D}$

$$\mathcal{D} = \Big[p_A(A), p_B(B)\Big]$$

Example 1 (Simple and Compound Lotteries) Consider the situation involving a player who has two possible moves T and B. If they choose T they receive \$100. If they choose B then they receive \$200 with probability 0.5 and \$0 with probability 0.5. Then based on the choice the player makes we get two lotteries of the outcomes :

$$\mathcal{D}_1 = \Big[\frac{1}{2}(\$200) + \frac{1}{2}(\$0)\Big]$$

$$\mathcal{D}_2 = \Big[\frac{1}{1}(\$100) + 0(\$0)\Big]$$

If we assume the player picks T with probability p_T and B with probability p_B, then we get a compound lottery

$$\widetilde{\mathcal{D}} = \Big[p_B(\mathcal{D}_1), p_T(\mathcal{D}_2)\Big]$$

Now instead of preferences over outcomes, we need to define preferences over lotteries or distributions. The main question addressed in this section, is how do you describe preferences with a utility function when the outcomes are stochastic i.e. given by some lottery. It turns out that if preferences satisfy certain conditions (which we call Von-Neumann-Morgenstern axioms), then preferences can be described by a linear utility function over a lottery $\mathcal{D}$.

Definition 4 (Linear Utility Function) A utility function $u : \Delta(\Omega) \to \mathbb{R}$ is linear if for every lottery $\mathcal{D} = \Big[p_1(A_1), \dots, p_K(A_K)\Big]$ given by

$$u(\mathcal{D}) = \sum_{\omega \in \Omega} p_\omega \cdot u(\omega) = \mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} u(X) \tag{2}$$

where $u(\omega)$ refers to the utility of outcome $\omega \in \Omega$. The term linear comes from the fact that $\sum_{\omega \in \Omega} p_\omega = 1$. For compound lotteries $\widetilde{\mathcal{D}} = [q_1(\mathcal{D}_1), \dots, q_K(\mathcal{D}_K)]$, we have the same idea but instead we use

$$u(\widetilde{\mathcal{D}}) = \sum_{k \in [K]} q_k \cdot u(\mathcal{D}_k) = \mathbb{E}_{X \xleftarrow{\$} \widetilde{\mathcal{D}}} u(\mathcal{D}_X) \tag{3}$$

Question 3 Which preference relations $\succeq$ can be represented by a linear utility function? To answer this question, we define 4 axioms called the Von-Neumann-Morgensterns Axioms, which guarantee that a utility function that can represent $\succeq$ with a linear utility function.

3 Von-Neumann-Morgensterns Axioms

Axiom 4 (The Archimedean/Continuity Axiom) For every triplet of outcomes $A \succeq B \succeq C$, there exists a scalar $\theta \in [0, 1]$ such that

$$B \sim [\theta(A) + (1 - \theta)(B)]$$

What does this say intuitively? Consider the situation where you get \$300 with probability 0.9 and \$0 with probability 0.1. Assuming more money is better, any reasonable person, would prefer this lottery to the lottery where we get \$300 with probability 0.1 and \$0 with probability 0.9. Consider another property of just receiving \$100 guaranteed. The above property is saying that preferences live on a continuous spectrum. That on side we have the first lottery where we win \$300 with high probability and on the other side we have the other lottery where we win nothing with high probability. For a certain probability (value of θ) on this spectrum, we will be indifferent between the deterministic outcome and the probabilistic outcome. Later we will see, that this indifference will be the point at which the expected returns are the same for both the lotteries.

Axiom 5 (Monotonicity) Let $\alpha, \beta \in [0, 1]$, such that $\mathcal{D}_1 = [\alpha(A), 1 - \alpha(B)]$, and, $\mathcal{D}_2 = [\beta(A), 1 - \beta(B)]$. Suppose, $A \succeq B$, then

$$\mathcal{D}_1 \succeq \mathcal{D}_2 \iff \alpha \geq \beta$$

What does this say intuitively? Quite simple really, I prefer that the outcome I like happens with higher probability. If I like rain, I prefer distributions with a higher probability of rain.

Theorem 5 For finite Ω, if $\succeq$ is complete, reflexive and transitive; and satisfies the archimedean axiom and monotonicity, and if $A \succeq B \succeq C$, then the value of θ defined in the continuity axiom is unique.

Proof. Exercise for the reader.

Hint: As $|\Omega|$ is finite, we are guaranteed the existence of a utility function by Theorem 2. Now use this utility function to prove the result. □

Axiom 6 (Simplification) For each $j = 1, \ldots, J$, let $\mathcal{D}_j$ be the following simple lottery

$$\mathcal{D}_j = \left[p_i^{(j)}(A_1), \ldots, p_K^{(j)}(A_K)\right] \tag{4}$$

Let $\widetilde{\mathcal{D}}$ be the following compound lottery:

$$\widetilde{\mathcal{D}} = \left[q_1(\mathcal{D}_1), \ldots, q_J(\mathcal{D}_J)\right] \tag{5}$$

For each $k = 1, \ldots, K$, define

$$r_k = \sum_{j=1}^{J} q_j p_k^{(j)} = \Pr_{X \xleftarrow{\$} \widetilde{\mathcal{D}}} [X = A_k] \tag{6}$$

as the probability of seeing the outcome A_k under $\widetilde{\mathcal{D}}$. Then we have

$$\widetilde{\mathcal{D}} \sim \mathcal{D}^* \tag{7}$$

where $\mathcal{D}^* = \Big[r_1(A_1), \dots, r_K(A_K)\Big]$

What does this say intuitively? The above axiom states that the only thing that affects preferences between lotteries is the probability distribution and NOT how the lotteries are conducted. As an example, consider $\mathcal{D}_1 = [\frac{1}{2}(A_1), \frac{1}{4}(A_2), \frac{1}{8}(A_3), \frac{1}{8}(A_4)]$. And consider the compound lottery $\widetilde{\mathcal{D}} = [\frac{3}{4}(\mathcal{D}_a), \frac{1}{4}(\mathcal{D}_b)]$, where $\mathcal{D}_a = [\frac{2}{3}(A_1), \frac{1}{3}(A_2)]$ and $\mathcal{D}_b = [\frac{1}{2}(A_3), \frac{1}{2}(A_4)]$. Although one lottery flips one coin, and the other makes use of two coins, the final probability of selecting A_i according to $\mathcal{D}$ or $\widetilde{\mathcal{D}}$ is the same for all $i \in [4]$. Thus the two lotteries are equivalent (or a player is indifferent between them).

Axiom 7 (Independence/Substiution) Let $\widetilde{\mathcal{D}} = \Big[q_1(\mathcal{D}_1), \dots, q_J(\mathcal{D}_J)\Big]$ be a compound lottery, and let $\mathcal{D}$ be a simple lottery. If $\mathcal{D}_j \sim \mathcal{D}$, then

$$\widetilde{\mathcal{D}} \sim \Big[q_1(\mathcal{D}_1), \dots, q_j(M), \dots, q_J(\mathcal{D}_J)\Big]$$

What does this say intuitively? The above axiom is sometimes called the axiom of substitution as it states we can swap between two indifferent lotteries.

4 Proving Von-Neumann-Morgensterns Theorem

Theorem 6 Assume a finite set of outcomes $\Omega = \{A_1, \dots, A_K\}$. If a player i's preference relation $\succeq$ over $\widetilde{\mathcal{D}}$ is complete, reflexive, transitive, and satisfies the four VNM axioms, if and only if the preference relation can be represented by a linear utility function.

Proof. The proof is given in the slides. Otherwise it can be found in [Rub12] as well. □

5 Attitude Towards Risk

In this section, we will make a certain set of assumptions that will allow us to look at a utility function and decide if the person associated with that utility function is risk-neutral, risk-seeking or risk-averse. The definitions for risk will only make sense under these assumptions.

Assumption 1 The set of outcomes $\Omega \subseteq \mathbb{R}$ is a subset of the reals. $x \in \Omega$ denotes the monetary reward a person receives. If Ω is finite then $\Omega = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Throughout this document we will assume that the set of outcomes is finite.

Note that the outome is **NOT** the utility for a person. For example, let $\Omega = [0, 10]$, then $x = 10$ just says that a person receives 10\$. The utility for a person describes how much 10\$ is worth to them. The utility function for person i will be described by a function $u_i : \Omega \to \mathbb{R}$.

Assumption 2 The utility function that represents players preference MUST be strictly monotonic. This is to say, no person, regardless of their utility function, will ever prefer an outcome $x \in \Omega$ over an outcome $y \in \Omega$ when $x < y$.

Assumption 3 We will assume that the players preferences over the outcomes satisfy the Von Nuemann-Morgenstern axioms.

By Assumption 3, for any lottery $\mathcal{D} \in \Delta(\Omega)$ with finite outcomes i.e $|\Omega| = k$, the utility of the lottery $\mathcal{D}$ must satisfy

$$u_i(\mathcal{D}) = \sum_{i=1}^{k} \Pr[\omega_i] u_i(\omega_i) \tag{8}$$

$$= \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u_i(x) \tag{9}$$

Definition 5 Player i is risk-neutral if for every lottery $\mathcal{D} \in \Delta(\Omega)$ with a finite number of possible outcomes

$$u_i(\mathcal{D}) = \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} x \tag{10}$$

Player i is risk-averse if for every lottery $\mathcal{D} \in \Delta(\Omega)$ with a finite number of possible outcomes

$$u_i(\mathcal{D}) \leq \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} x \tag{11}$$

Player i is risk-seeking (risk-loving) if for every lottery $\mathcal{D} \in \Delta(\Omega)$ with a finite number of possible outcomes

$$u_i(\mathcal{D}) \geq \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} x \tag{12}$$

One way to check if a player is risk-averse is to enumerate if every lottery in $\mathcal{D} \in \Delta(\Omega)$ and check the above conditions. This might be computationally infeasible, so instead we show that suffices to check two arbitrary outcomes.

Theorem 7 For every $p \in [0, 1]$ and every pair of outcomes $x, y \in \Omega$, where $\mathcal{D} = (\Pr[x], \Pr[y]) = (p, 1-p)$ the following holds:

1. Player i is risk-neutral if and only if

$$u_i(\mathcal{D}) = \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u_i(x) = u_i(p * x + (1-p) * y) \tag{13}$$

2. Similarly, player is risk-averse if and only if

$$u_i(\mathcal{D}) = \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u_i(x) \leq u_i(p * x + (1-p) * y) \tag{14}$$

3. And risk-seeking if and only if

$$u_i(\mathcal{D}) = \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u_i(x) \geq u_i(p * x + (1-p) * y) \tag{15}$$

Proof. The proof of this theorem follows trivially from the next theorem. □

It turns out, there is an even easier method to check if a person is risk-averse or risk-seeking. First we need to some definitions from convex analysis (see [AA16] for a quick refresher).

Definition 6 (Convex Function) We say that a function $f : \Omega \to \mathbb{R}$ is convex if for every $x, y \in (a, b)$ and every $\lambda \in (0, 1)$

$$f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda)f(y)$$

Definition 7 (Concave Function) We say that a function $f : \Omega \to \mathbb{R}$ is concave if for every $x, y \in (a, b)$ and every $\lambda \in (0, 1)$

$$f\Big(\lambda x + (1-\lambda)y\Big) \geq \lambda f(x) + (1-\lambda)f(y)$$

Corollary 1 (Linear Function) A function is linear if it is both convex and concave.

Theorem 8 (Jensen's Inequality) If f is a convex function on Ω and $\mathcal{D} \in \Delta(\Omega)$ is a random variable taking values in Ω, then

$$f(\mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} X) \leq \mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} f(X)$$

If f is concave then

$$\mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} f(X) \leq f(\mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} X)$$

Theorem 9 A player i whose preference relations satisfy the Von-Neumann-Morgenstern axioms

1. is risk averse if and only if u_i is a concave function.
2. is risk seeking if and only if u_i is a convex function.
3. is risk-neutral if and only if u_i is a linear function.

Proof. Let $\mathcal{D} = \Big(\Pr[x_1], \dots, \Pr[x_k]\Big) = (p_1, \dots, p_k)$. We have

$$u_i(\mathcal{D}) = \sum_{i=1}^{k} p_i u_i(x_i)$$

We prove the theorem for risk averse, and the rest are identical. Assuming the person is risk-averse, we want to show that u_i is concave. From Defintion 5 we have a player is risk averse has $u_i(\mathcal{D}) \leq u_i(\mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} x)$ for any $\mathcal{D} \in \Delta(\Omega)$.

$$u_i(\mathcal{D}) = \mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} u_i(x)) = \sum_{i=1}^{k} p_i \cdot u_i(x_i) \leq u_i\Big(\mathbb{E}_{x \xleftarrow{\$} \mathcal{D}} x\Big) = u_i\left(\sum_{i=1}^{k} p_i \cdot x_i\right) \tag{16}$$

As $\sum_{i=1}^{k} p_i = 1$, from the definition of a concave function[1] (Definition 7) we have that u_i is concave. Now assume that u_i is concave, then, we get what we want by Jensens Inequality we have

$$\mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} u_i(X) \leq u_i\left(\mathbb{E}_{X \xleftarrow{\$} \mathcal{D}} X\right)$$

[1]The definition I have given is not exactly the definition that I use in (16). However, try and prove that the definition and the above statement are equivalent. Hint: Use induction. Use https://math.stackexchange.com/questions/3307227/for-f-convex-and-lambda-i0-sum-lambda-i-1-x-i-in-mathbbr-does if you get stuck.

The risk-seeking proof is identical with the inequalities switched. For risk-neutral we use the two proofs and invoke 1. ☐

References

[AA16] G. Hall A.A. Ahmadi. Lecture 7. https://www.princeton.edu/~aaa/Public/Teaching/ORF523/S16/ORF523_S16_Lec7_gh.pdf, 2016.

[MZS20] Michael Maschler, Shmuel Zamir, and Eilon Solan. *Game theory*. Cambridge University Press, 2020.

[Rig18] Luca Rigotti. Advanced microeconomic theory 1, lecture 3:preference and utility. https://sites.pitt.edu/~luca/ECON2100/2018Class/lecture_03.pdf, 2018.

[Rub12] Ariel Rubenstein. Lecture notes in microeconomic theory. https://arielrubinstein.tau.ac.il/books/PUP2020.pdf, 2012.