PROBLEM 5

Given two multi-sets of votes U and V over the same candidate set, let $d_M(U,V)$ be the min number of votes that need to be added or deleted from U to transform it into V.

Argue that $d_M(\ ,\ )$ is a distance metric.

- Reflexivity: $d_M(U,U) = 0$ By defn, don't need to do anything to change U to U.
- Positivity: $d_M(U,V) > 0$ if $U \neq V$ (similar reason ⤴)
- $\triangle$ inequality: Transform $U \to W$ then transform $W \to V$.

$$\therefore\ d(U,V) \leq d(U,W) + d(W,V)$$

(Either W in the shortest transformation from U to V, then equality or W is a long winded transformation so more work than needed)

- Symmetry: $d_M(U,V) = x+y$

  x deletions y insertions

  $d_M(V,U) = x+y$ switch order

  $= x$ insertions y deletions

$$\therefore\ d_M(U,V) = d_M(V,U)$$

Given a profile R, let $\tilde{M}(a)$ be the min number of voters that can be added to R to make a condorcet winner.

Argue that for any $a,b \in C$ we have

$$m(a) < m(b) \iff \tilde{M}(a) > \tilde{M}(b)$$

where $m(a)$: maxmin score of a.

↳ This is just the # times a beats their toughest rival (one who beats them the most times)

$$m(a) = \min_{b \in C \setminus \{a\}} \left|\{v_i \in R : a \succ_i b\}\right|$$

$\boxed{\text{Soln}}$ Fact: Adding $\tilde{M}(a) = n - 2m(a) + 1$ voters who rank a first ensures a is a condorcet winner.

Why? Observe $N = n + n - 2m(a) + 1 = 2n - 2m(a) + 1$.

cos m(a) is a measure of toughest opponent.

Prior to this addition a was beating every other player by at least m(a) votes in a head to head.

Now a beats every player by at least : $m(a) + n - 2m(a) + 1 = \frac{N+1}{2}$ votes.

Now if $M(a) < M(b) \Rightarrow n - 2m(a) + 1 < n - 2m(b) + 1$

$\therefore m(b) < m(a)$.

if $m(a) > m(b) \Rightarrow M(b) > M(a)$ [just plug in above equations]

This assumes neither a or b are condorcet winners.

But even when a is a condorcet winner but b is NOT; it follows trivially:

$M(a) = 0$

$M(b) > 0 \quad \therefore M(a) < M(b)$

We have $n - m(\alpha) < \frac{n+1}{2}$ [as a is a condorcet winner]

$n - m(b) > \frac{n+1}{2}$ [b is NOT a c.w so some player beats b $> (n+1)/2$ times]

$\therefore n - m(\alpha) < n - m(b)$ or $m(b) < m(a)$!

Use the results of the previous parts to provide a distance rationalisation of the maxmin rule w.r.t to the condorcet consensus.

This is a lot of fancy words to saying the following. You are given some multiset $\mathcal{U}$ and a scoring rule $R : \underbrace{C}_{\text{candidates}} \times \underbrace{\mathcal{V}}_{\text{voting profiles}} \rightarrow \underbrace{\mathbb{R}}_{\text{Reals}}$. In this case the scoring rule is maxmin.

You want to find a class of profiles (such as profiles w/ condorcet winner) and some distance metric on profiles s.t the following process yields the same winner as $R(\mathcal{U})$.

$$\left\{ \text{Cond Winner}(v^*) : v^* = \underset{v \in \underbrace{Cls}_{\text{only iterate through condorcet winner profiles.}}}{\text{argmin}} d(\mathcal{U}, v) \right\}$$

Now given a profile V, $\alpha \in C$ is maxmin winner.

Then if $V \in CLS$ and α is cond winner at V, then $d_M(V, V) = 0$ and $d_M(V, W) > 0$ for $W \in CLS$ for which α is not winner.

So α will be the winner of argmin process.

Now suppose α is not a condorcet winner @ V.

To make α a condorcet winner we can find profile V' by adding $\underbrace{n - 2m(a) + 1}_{\tilde{M}(\alpha)}$ voters who rank α as top. to V; where n is num voters in V.

Now suppose there is another candidate β that is not maxmin winner @ V (so $m(b) < m(a)$)

but we can add $\tilde{M}(b)$ votes to make β cond winner.

By result above we have $\tilde{M}(b) > \tilde{M}(\alpha)$ since $m(b) < m(a)$.

Therefore the profile V'' you get to make b a winner

is further away from V' i.e $d_M(V, V') < d_M(V, V'')$.

So the min process will always pick α as cond winner!

Why did we use multi-sets instead of profiles.

It makes the defn of distance more painful.

Have to account for order of deletions and insertions.

Why use both insertions and deletions to defined d_M?

* makes symmetry easier when one set $U \subseteq W$.

* could we only insertions but change

the defn of d_M to min votes I need to add

to both U & W to make the same.

* a bit more painful.!

PROBLEM 4

Show that $d_{SWAP}(U,V) = \sum_{i=1}^{n} d_{swap}(u_i, v_i)$ is a distance metric! when U, V are profiles of the same size

One line proof: Lin comb of distance metrics are dist metrics

- Symmetry: $d_s(U,V) = \sum_{i=1}^{n} d_s(u_i,v_i)$
  $= \sum_{i=1}^{n} d_s(v_i,u_i)$ as d_s is a metric!
  $= d_s(V,U)$
- Reflexivity: $d_s(U,U) = \sum_{i=1}^{n} d_s(u_i,u_i) = 0$
- Positivity: $d_s(U,V) = \sum_{i=1}^{n} d_s(u_i,v_i) > 0$, $V \neq U$
  Must exist at least one $j \in [n]$ s.t. $d_s(u_j, v_j) > 0$ otherwise $U = V$.
- Δ inequality: $d_s(U,W) = \sum_{i \in [n]} d_s(u_i,v_i) \leq \sum_{i \in [n]} d_s(u_i,w_i) + d(w_i,v_i)$
  $= d_s(U,W) + d_s(W,V)$

PROBLEM 3

Show that when num voters is odd and V is single peaked on a tree $T = (C,E)$; then we always have a condorcet winner.

ASIDE: See posted notes on what it means to be single peaked on a tree!

Here by tree we mean an undirected, connected graph without any cycles.

Now take your tree and convert it into a DAG. $G = (C, E')$

That is

For any $a,b \in C$ we have $a \mapsto b \in E' \iff (a,b) \in E$ (original Tree) and a beats b by a majority of votes.

As original tree didn't have cycles, we know that the DAG can't have cycles.

So we have a node $c \in G$ that is a source node.

We claim the c is a condorcet winner

For any $d \in C \setminus \{c\}$; if $c \to d$ i.e d is a direct nbr of c; the c beats d head to head. For d that is not a nbr of c we have $c \to x \to \cdots d$.

But if $c \to x$ then we must have $x \succ_{maj} d$., so $c \succ_{maj} d$

Why? Assume you have

$$c \to x \leftarrow b$$

i.e $c \succ_{majority} x$ and $b \succ_{maj} x$; then this breaks single-peakedness.

more than half the people prefer c over x.

By single peakedness they cannot prefer b over x as b is further away from c than x which is their most preferred out of the 3.

Problem 2 $P = (v_1 \dots v_n)$ all votes are distinct.

Argue there are at most 2 ways to make P single crossing!

Soln: If there is no way to permute P then we win.

Assume there is a permutation of P and denote it as $(v_1, v_2 \cdots - v_n.)$ that is single crossing.

Then $(v_n, \dots v_1)$ is also single crossing!

We claim that these are the only two ways to make it single crossing.

Proof by induction

Base case: $n \leq 2$. (v_1, v_2) and (v_2, v_1) are the only permutations! Done.

Induction Hypothesis : Assume for size $n-1$; there are at most 2 ways to make single crossing profiles.

Induction step: Let $P = (v_1 \ldots v_n)$; As all v_i's are distinct

$v_1 \neq v_2$ i.e $\exists\, \alpha, \beta$ where $\alpha \succeq_1 \beta$ and $\alpha \preceq_2 \beta$.
But as P is single crossing for $j \geq 2$ we must have $\alpha \preceq_j \beta$ otherwise α, β cross again.
So v_1 must be at the extremes of any valid ordering.

So my options are $v_1 \ldots v_n$ #
$v_n \ldots v_1$ ##
* $v_1 \boxed{v_n \ldots v_2}$
** $\boxed{v_2 \ldots v_n}\, v_1$

single crossing

Want to show that these are not single crossing

Now as P is in single crossing; so is $P' = (v_2, v_3 \ldots v_n)$; and there are 2 ways at most to make P' single crossing (by induction hypothesis)

Since $v_2 \neq v_n$ $\exists\, c, d$ s.t $c \succ_2 d$ and $d \succ_n c$.
If P were to be single crossing then we need
$c \succ_1 d$ (otherwise # and ## are not single crossing and we assumed # was single ")

But if $c \succ_1 d$ then c and d cross over twice in

$v_2 \ldots v_n\, v_1$ and

$v_1\, v_n \ldots v_2$

So these perms cannot be single crossing.

Problem 1 Prove that every $P = (v_1 \dots v_n)$ over m candidates is single peaked wrt to atmost 2^{m-1} axes

P is single peaked w.r.t. some axis Δ if every $v_i \in P$ is single peaked w.r.t to Δ. Thus show that $\exists$ some v_i st v_i is single peaked w.r.t at most 2^{m-1} axes.

By induction.

$m = 1$. Only one way to rank candidates and be SP. $2^{1-1} = 1$.

Induction Hypothesis: For $m-1$; there are 2^{m-1} axes $\Delta_1, \dots \Delta_{2^{m-1-1}}$

Induction step: Pick some voter ranking v_i

$v_i = a > b \dots > m$.
Create v_i' by deleting m.
v_i' is a ranking over 2^{m-1-1} axes.

if I put m at the extreme left for any Δ_j $j \in [2^{m-2}]$
$m \angle \Delta_j$ is also single peaked.
Similarly placing it on the extreme right also makes
v_i SP w.r.t $\Delta_j \angle m$.

Thus 2 ways to extend each Δ_j.

Total $= 2 \cdot 2^{m-1-1} = 2^{m-1}$!