



Now if  $M(a) < M(b) \Rightarrow n - 2m(a) + 1 < n - 2m(b) + 1$

$\therefore m(b) < m(a)$ .

if  $m(a) > m(b) \Rightarrow M(b) > M(a)$  [just plug in above equations]

This assumes neither a or b are Condorcet winners.

But even when a is a Condorcet winner but b is NOT; it follows trivially:

$M(a) = 0$

$M(b) > 0 \therefore M(a) < M(b)$

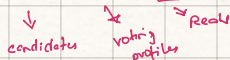
We have  $n - m(a) < \frac{n+1}{2}$  [as a is a Condorcet winner]

$n - m(b) > \frac{n+1}{2}$  [b is NOT a c.w so some player beats b  $> (n+1)/2$  times]

$\therefore n - m(a) < n - m(b)$  or  $m(b) < m(a)$  !

Use the results of the previous parts to provide a distance rationalisation of the maxmin rule w.r.t to the Condorcet consensus.

This is a lot of fancy words to saying the following. You are given some multiset  $U$  and a scoring rule  $R: C \times \mathcal{V} \rightarrow \mathbb{R}$ . In this case the scoring rule is maxmin.



You want to find a class of profiles (such as profiles w/ Condorcet winner) and some distance metric on profiles s.t the following process yields the same winner as  $R(U)$ .

$\{ \text{Cond Winner } (v^*) : v^* = \underset{v \in \text{CLS}}{\text{argmin}} d(U, v) \}$

↓  
only iterate through Condorcet winner profiles.

Now given a profile  $v$ ,  $d \in C$  is maxmin winner.

Then if  $v \in \text{CLS}$ ; then  $d_M(v, v) = 0$  and  $d_M(v, w) > 0$  for  $w \in \text{CLS}$  for which  $d$  is not winner.

So  $d$  will be the winner of argmin process.

Now suppose  $a$  is not a Condorcet winner @  $V$ .

To make  $a$  a Condorcet winner we can find profile  $V'$  by adding  $\underbrace{n - 2m(a) + 1}_{\tilde{M}(a)}$  voters who rank  $a$  as top. to  $V$ ; where  $n$  is num voters in  $V$ .

Now suppose there is another candidate  $B$  that is not maxmin winner @  $V$  (so  $m(b) < m(a)$ ) but we can add  $\tilde{M}(b)$  votes to make  $B$  cond winner.

By result above we have  $\tilde{M}(b) > \tilde{M}(a)$  since  $m(b) < m(a)$ .

Therefore the profile  $V''$  you get to make  $b$  a winner is further away from  $V'$  i.e.  $d_M(V, V') < d_M(V, V'')$ .

So the min process will always pick  $a$  as cond winner!

Why did we use multi-sets instead of profiles.

It makes the defn of distance more painful.

Have to account for order of deletions and insertions.

Why use both insertions and deletions to defined  $d_M$ ?

\* makes symmetry easier when one set  $U \subseteq W$ .

\* could use only insertions but change

the defn of  $d_M$  to min votes I need to add to both  $U$  &  $W$  to make the same.

\* a bit more painful!

### PROBLEM 4

Show that  $d_{\text{SWAP}}(U, V) = \sum_{i=1}^n d_{\text{SWAP}}(u_i, v_i)$  is a distance metric!  
where  $U, V$  are profiles  
of the same size

• Symmetry:  $d_g(U, V) = \sum_{i=1}^n d_g(u_i, v_i)$   
 $= \sum_{i=1}^n d_g(v_i, u_i)$  as  $d_g$  is a metric!  
 $= d_g(V, U)$

• Reflexivity:  $d_g(U, U) = \sum_{i=1}^n d_g(u_i, u_i) = 0$

• Positivity:  $d_g(U, V) = \sum_{i=1}^n d_g(u_i, v_i)$  Must exist at least one  
 $V \neq U > 0$   $j \in [n]$  st  $d_g(u_j, v_j) > 0$   
otherwise  $U = V$ .

•  $\Delta$  inequality:  $d_g(U, W) = \sum_{i \in [n]} d_g(u_i, v_i) \leq \sum_{i \in [n]} d_g(u_i, w_i) + d(w_i, v_i)$   
 $= d_g(U, W) + d_g(W, V)$

One line proof: Lin comb of  
distance metrics  
are dist metrics

### PROBLEM 3

Show that when num voters is odd and  $V$  is single peaked on  
a tree  $T = (C, E)$ ; then we always have a Condorcet winner.

ASIDE: See posted notes on what it means to be single peaked on a tree!

Here by tree we mean an undirected, connected graph without any cycles.

Now take your tree and convert it into a DAG.  $G = (C, E')$

That is

For any  $a, b \in C$  we have  $a \rightarrow b \in E' \Leftrightarrow (a, b) \in E$  and  $a$  beats  $b$  by  
 $\uparrow$   
original tree a majority of  
votes.

As original tree didn't have cycles, we know that the DAG can't have cycles.

So we have a node  $c \in G$  that is a source node.

We claim the  $c$  is a Condorcet winner

For any  $d \in C \setminus \{c\}$ ; if  $c \rightarrow d$  i.e.  $d$  is a direct nbr of  $c$ ; then  $c$  beats  $d$  head to head. For  $d$  that is not a nbr of  $c$  we have  $c \rightarrow x \rightarrow \dots \rightarrow d$ .

But if  $c \rightarrow x$  then we must have  $x \succ_{\text{maj}} d$ , so  $c \succ_{\text{maj}} d$

Why? Assume you have  
 $c \rightarrow x \leftarrow b$

i.e.  $c \succ_{\text{maj}} x$  and  $b \succ_{\text{maj}} x$ ; then this breaks single-peakedness.

more than half the people prefer  $c$  over  $x$ .

By single peakedness

they cannot prefer  $b$  over  $x$  as  $b$  is further away from  $c$  than  $x$  which is their most preferred out of the 3.

**Problem 2**  $P = (v_1, \dots, v_n)$  all votes are distinct.

Argue there are at most 2 ways to make  $P$  single crossing!

**Soln:** If there is no way to permute  $P$  then we win.

Assume there is a permutation of  $P$  and denote it as  $(v_1, v_2, \dots, v_n)$  that is single crossing.

Then  $(v_n, \dots, v_1)$  is also single crossing!

We claim that these are the only two ways to make it single crossing.

Proof by induction

Base case:  $n \leq 2$ .  $(v_1, v_2)$  and  $(v_2, v_1)$  are the only permutations! Done.

Induction Hypothesis: Assume for size  $n-1$ ; there are at most 2 ways to make single crossing profiles.

Induction step: let  $P = (v_1 \dots v_n)$ ; As all  $v_i$ 's are distinct

$v_1 \neq v_2$  i.e.  $\exists \alpha, \beta$  where  $\alpha \geq \beta$  and  $\alpha < \beta$ .  
 But as  $P$  is single crossing for  $j \geq 2$  we must have  $\alpha < \beta$  otherwise  $\alpha, \beta$  cross again.  
 So  $v_1$  must be at the extremes of any valid ordering.

So my options are  $v_1 \dots v_n \#$   
 $v_n \dots v_1 \#\#$

single crossing } want to show that  
 $\# v_1 v_n \dots v_2$  } these are not single crossing  
 $\#\# v_2 \dots v_n v_1$  }

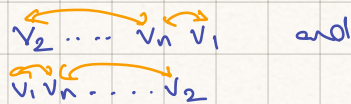
Now as  $P$  is in single crossing; so is  $P' = (v_2, v_2 \dots v_n)$ ; and there are 2 ways at most to make  $P'$  single crossing (by induction hypothesis)

Since  $v_2 \neq v_n \exists c, d$  s.t.  $c \geq d$  and  $d \geq c$ .

If  $P$  were to be single crossing then we need

$c \geq d$  (otherwise  $\#$  and  $\#\#$  are not single crossing and we assumed  $\#$  was single " )

But if  $c < d$  then  $c$  and  $d$  cross over twice in



So these perms cannot be single crossing.

**Problem 1**

Prove that every  $P = (v_1, \dots, v_n)$  over  $m$  candidates is single peaked w.r.t. to at most  $2^{m-1}$  axes.

$P$  is single peaked w.r.t. some axis  $\Delta$  if every  $v_i \in P$  is single peaked w.r.t. to  $\Delta$ . Thus show that  $\exists$  some  $v_i$  st  $v_i$  is single peaked w.r.t. at most  $2^{m-1}$  axes.

By induction.

$m = 1$ . Only one way to rank candidates and be SP.  
 $2^{1-1} = 1$ .

Induction Hypothesis: For  $m-1$ ; there are  $2^{m-1}$  axes  $\Delta_1, \dots, \Delta_{2^{m-1}}$

Induction step: Pick some voter ranking  $v_i$

$v_i = a > b \dots > m$ .

Create  $v'_i$  by deleting  $m$ .

$v'_i$  is a ranking over  $2^{m-1-1}$  axes.

if I put  $m$  at the extreme left for any  $\Delta_j$   $j \in [2^{m-1}]$   
 $m \triangleleft \Delta_j$  is also single peaked.

Similarly placing it on the extreme right also makes  $v_i$  SP w.r.t.  $\Delta_j \triangleleft m$ .

Thus 2 ways to extend each  $\Delta_j$ .

Total =  $2 \cdot 2^{m-1-1} = 2^{m-1}$ !

