Problem 8 Prove that weighted coalition manipulation under veto rule is NP-complete!

Soln: Let $m=3$; Candidates a, b, p.

$t+1$ voters; t of them form a coalition

The first guy has weight $2k-1$

↳ these guys have weights $2x_1, \dots, 2x_t$

Want to reduce this problem to an NP complete problem; in particular PARTITION.

Assume that the first guy ranks p LAST. Our coalition group wants to make p win. Partition $1, \dots, t$ into 2 groups S and S' s.t $\underbrace{\sum_{i \in S} x_i = 2k}_{\text{This group ranks } a \text{ last}}$ and $\underbrace{\sum_{i \in S'} x_i = 2k}_{\text{This group ranks } b \text{ last.}}$

This guarantees p beats a & b ← (write down the scores like on the slides; fairly obvious)

So if this coalition can get p to uniquely win; then this coalition can solve coalition!

PROBLEM 7 Manipulating votes with YES/NO!

Each i votes for $S \subseteq C$ and are indifferent between them.

Let $R = (v_1, \dots v_n)$ be a voter profile. The voting rule is manipulable if $\exists v_i'$ s.t under R player i disapproves of the winner of $F(R)$ but approves $F(R')$ where $R' = (v_1 \dots v_{i-1}, v_i', v_{i+1}, \dots v_n)$.

Assume i can manipulate voting rule F. Let a be the winner of $F(R)$.

i does not approve of a. So there is nothing i can do to decrease a's approval score. It also can't bump up another candidate they approve of; as they already included them in their approval set in R.

They can only include someone they don't approve of but if they win its pointless.

Problem 6

F is strongly monotone $\Leftrightarrow$ F is not manipulable.

$\hookrightarrow$ Every Ranking $R \in L(C)^n$

Every voter $i \in [n]$

Every ranking $v_i' \in L(c) \longrightarrow$ the following holds: Set R' by swapping v_i with v_i' in R.

if $F(R) \neq F(R') \Rightarrow$ in v_i candidate $F(R')$ appears below $F(R)$ and in v_i' $F(R')$ appears above $F(R)$.

$\Rightarrow$ Assume that F is manipulable; want to show F is not monotone.

$\exists$ a profile $R = (v_1, \dots, v_n)$ s.t i can swap v_i in R to with v_i' to get R' s.t $F(R') > F(R)$ in v_i. This violates $\xrightarrow[\text{monotoncity}]{}$ which states $\forall R, \forall i, \forall v_i'$ v_i: $F(R) > F(R')$ & v_i': $F(R') > F(R)$

$\Leftarrow$ Assume F is not monotone. Want to show F is manipulable. $\longrightarrow \exists R$ s.t $R' = (v_1, \dots v_i' \dots v_n)$ where v_i: $F(R') > F(R)$ or v_i': $F(R) > F(R')$

So either player i can manipulate R (play v_i' in R) or R' to get what they want. (play v_i in R')

PROBLEM 4

If $S \subseteq [n]$ is F-decisive for $(x,y) \in (C \times C)$ then it is F-decisive for all pairs (z, y) with $z \in C \setminus \{x, y\}$.

F-decisive:
S is F-decisive if $\forall i \in S$; $a > b$. then $a > b$ in the Final ranking $F(R)$

Consider a profile where all voters in S rank z above y.

For all $i \in S$: $z > x > y$

$i \notin S$: $z > X > y$

$\}$ modified profile

By II-A: This does not change the order of z and y in the final outcome.

As S is weakly F decisive: x appears before y in final output.

By pareto efficiency: z above x; $\therefore z$ above y by transitivity.

By IIA : This is also True for the unmodified profile

↳ "If a is ranked above b in current profile and we permute candidates w/o changing relative order of a and b, then a will still be ranked above b in the resulting profile.

PROBLEM 3

Why does the dictator rule satisfy the IIA axiom?

Let's say dictator i prefers candidate a over b.

Then regardless of how you permute his ranking and other voter rankings, as long as everyone prefers a over b; then i will still prefer a over b. And a is ranked before b.

PROBLEM 2

<u>Claim:</u> $|B(u,v)| \geq |S(u,v)|$

To go from ranking $u \to v$; we have to swap all pairs $a, b \in C$ that disagree in u and v.

if $\alpha, \beta \in S(u,v) \Rightarrow$ They have to be swapped at some pt.

$\therefore \; S(u,v) \subseteq B(u,v)$

For every swap I can find a pair

#swaps cannot exceed S(u,v) → I always have room

<u>Claim</u> : $B(u,v) \leq S(u,v)$.

Let $v : 1 \succ 2 \succ \ldots m$

so $B(u,v)$ is # of swaps needed to sort u in ascending order.

So for each swap of adjacent candidates, we lower $S(u,v)$ by 1

We will show that if $u \neq v$; $\exists (a,b) \in u$ that are adjacent and in descending order.

j is the first candidate not in the same position in u & v. Say position ℓ in u.

1, 2, 3, $\boxed{4}$

1, 2, 3, 7, 8, $\boxed{4}$

<u>swap.</u>

PROBLEM 1

- Symmetry

$$S(u,v) = \{ \{a,b\} : a \succsim_u b \wedge b \succsim_v a \}$$
$$= \{ \{b,a\} : b \succsim_v a \wedge a \succsim_u b \}$$
$$= S(v,u)$$
$$\therefore |S(u,v)| = |S(v,u)|$$

- Positivity

$S(u,v) > 0$ if $u \neq 0$; at least one disagreement.

- Reflexivity

$d(u,u) = 0$ # can't disagree on any pair. Two rankings are same.

- Δ - inequality :

Pick $w \in L(C)$; Pick $\{a,b\} \in S(u,v)$

either $a \succsim_w b$ or $b \succsim_w a$

$\downarrow$ $\{a,b\} \in S(w,v)$ as $a \succsim_w b$, $b \succsim_v a$

$\rightsquigarrow$ $\{a,b\} \in S(u,w)$ $b \succsim_w a$, $a \succsim_u b$.

Thus $\{a,b\} \in S(w,v) \cup S(u,w)$.

$$\therefore |S(u,v)| \leq |S(u,w)| + |S(w,v)|$$

Problem 5

Monotonicity : if i bumps c up; c's ranking cannot go down!