Problem 1a

$G = [13; 7, 7, 3, 3]$ $2\phi_7 + 2\phi_3 = 1$ so we can just compute the shapley value for any one player.

I only need to consider winning permutations where 7 adds value to the game. Pick player 7.

7 adds value when ① 7 is third $\boxed{\ \ \ \ \ 7 \ }$ 3 slots 3 options : $\boxed{3!}$ ways

② 7 is second but preceded by 7 $\boxed{7\ 7\ \ \ }$ $\boxed{2}$ ways to arrange the 3's.

$$\phi_7 = (3! + 2!)/4! = 1/3$$

$$\phi_3 = (1 - 2\phi_7)/2 = 1/6$$

Problem 1b

$G = [10; 3, 3, 3, 3, 1, 1]$ 1 must be fourth to be influential.

Need 3 3's here out of 4 $\therefore \binom{4}{3} 3!$

$$\phi_1 = \binom{4}{3} \cdot 3! \cdot 2/6! = 1/15$$

$$\phi_3 = 13/60$$

2 ways of ordering 3 and 1.

Problem 2

Given a weighted voting game $G = [q; w_1, \dots w_n]$. Prove that $w_i \le w_j \Rightarrow \phi_i(G) \le \phi_j(G)$

Let Π be a permutation for which player i is pivotal.

Let $\Pi_{i \leftrightarrow j}$ be the permutation obtained by swapping i and j.

Let S be the predecessors of i in Π $\underbrace{\boxed{\pi_1 \mid \pi_2 \mid \dots}}_{S}\boxed{i \mid \ \ }$. As i is pivotal we have $w(S) < q$ but $w(S \cup \{i\}) \ge q$.

Now imagine $j \notin S$. Then $w(S \cup \{j\}) \ge w(S \cup \{i\}) \ge q$

$j \in S$ Then $w(S \setminus \{j\} \cup \{i\}) = w(S) - w_j + w_i < q$

Adding j to $S \setminus \{j\} \cup \{i\}$ is pivotal. $\therefore$ j is pivotal to S.

Observe that $\Pi \ne \Pi'$; then $\Pi_{i \leftrightarrow j} \ne \Pi'_{i \leftrightarrow j}$; and j will be pivotal for the set S' that are predecessors of i in Π' (where i is pivotal)

Therefore the set of permutations where j is pivotal is at least as large as the set of permutations i is pivotal in. i.e $\phi(i) \le \phi(j)$

Problem 4

Can you compute Banzhaf index in poly time? What about pseudo poly time?

No! To set banzhaf index of null player i equal to $\phi_i = 0$, we need to check if a player is not null in a weighted voting game, which we know is NP-hard.

So unless $P = NP$; it is not possible to compute the banzhaf index in poly time in the size of the instance.

Pseudopoly time? Yes → Use dynamic programming.

$$\boxed{\beta_n(G) = \frac{1}{2^{n-1}} \sum_{w = q - w_n}^{q-1} Z(w)}$$

→ number of subsets of $[n]$ that have weight w!

Define $U(w,k) = \left| \{ S \subseteq \{1,2 \dots k\} : \sum_{x \in S} w_x = w \} \right|$

number of subsets of $[k]$ that have value w

$Z(W) = U(w, n-1)$; Recursive Equation: $U(w,k) = \underbrace{U(w,k-1)}_{\text{Player } k \text{ has no impact}} + \underbrace{U(w-w_k, k-1)}_{\text{player } k \text{ gets the subset upto } k.}$; $U(w,1) = \begin{cases} 1 & \text{if } w = w_1 \text{ or } w = 0 \\ 0 & \text{otherwise} \end{cases}$ Base case

Problem 7

Compute the Schultz winner in poly time?

Suffices to compute $p[a,b]$ for all $a,b \in C$.

shortest paths in majority graph

To compute $p[a,b]$

Iterate k in $[\frac{n}{2}, \frac{n}{2}+1, \dots n]$ // could do with binary search

* construct k-majority graph
* output largest k for which there is a path from a to b.

do it for all pair $a,b \in C$ and then find a st $p[a,b] \geq p[b,a]$

* path checking $O(m)$
* search $O(n)$ or $O(\log n)$ $\longrightarrow$ Total complexity $O(m^3 \log n)$
* # pairs $O(m^2)$

Problem 3

Game $G = (N, v)$; Show that v can be expressed as a linear combination of unanimity games.

$$\boxed{v \equiv \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) u_S}$$

Fix some coalition Q. For $S \subseteq N$; $u_S(Q) = 1$ if $S \subseteq Q$ 0 otherwise.

Want to show:

$$\boxed{v(Q) = \sum_{\substack{S \subseteq Q \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) u_S(Q)}$$

$$\sum_{\substack{S \subseteq Q \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) \cdot 1$$

Re-write as

$$\boxed{\sum_{T \subseteq Q} \sum_{S : T \subseteq S \subseteq Q} \dots}$$

CASE I: If $T = Q$, then $S = T = Q$.

$(-1)^0 v(Q) \cdot = v(Q)$

CASE II: If $T \subset Q$, let $|T| = t$ $|Q| = q$; we have $q-t$ elements in Q that are not in T.

We have $\binom{q-t}{s}$ ways of making a set S of size $t+s$. s.t $T \subseteq S$ and $T \subset Q$. $\therefore |S| - |T| = s$; T contributes

$$\binom{q-t}{s}(-1)^s v(T)$$

when $|S| = t+s$.

Now s goes from $0, 1 \dots, (q-t)$ (If s was any bigger T would contain some elements of Q)

$\therefore$ T's total contribution is $\boxed{\sum_{s=0}^{q-t} \binom{q-t}{s} (-1)^s v(T)}$ } Taylor Series / Binomial thm / whatever you call this by

$= (1 + (-1))^{q-t} = 0$

PROBLEM 5

Show that the Borda count winner cannot be a condorcet loser.

We have m candidates $C = \{1, \dots m\}$ and n voters.

Total score for all candidates $\underbrace{n}_{\text{n voters vote once for every position.}} \cdot (1+2+\dots m-1) = n \cdot \frac{m(m-1)}{2}$. By the pigeon hole principle the winners score must be $\geq \frac{n(m-1)}{2}$ ##

FACT: For a fixed voter the Borda count of a candidate is the number of candidates ranked lower than the candidate.

$\therefore$ Borda score of candidate $b = \sum_{i \in [n]} \sum_{a \in C \setminus \{b\}} \mathbb{1}[a \text{ is ranked lower than } b]$

Indicator fxn.

$$= \sum_{a \in C \setminus \{b\}} \underbrace{\boxed{\sum_{i \in [n]} \mathbb{1}[a \text{ is ranked lower than } b]}}_{\text{Number of times } b \text{ beats } a \text{ in head to head battles}}$$

Assume b is a condorcet loser. b loses every head to head game (all $m-1$ one of them) ie the total votes b gets is $< \frac{n}{2}(m-1)$

which contradicts ##.

Problem 6

$s_1 = 3$	Num votes					
	3	c_1	$\succeq$	c_2	$\succeq$	c_3
$s_2 = 2$	2	c_2	$\succeq$	c_1	$\succeq$	c_3
$s_3 = 2$	2	c_3	$\succeq$	c_2	$\succeq$	c_1

$T_1 = \{c_1\}$ $T_2 = \{c_2, c_3\}$

$P = \{\{c_1, c_2\}, \{c_1, c_3\}\}$

EARLY: Pit c_1 and c_2 : c_2 wins

LATE: Pit (c_1, c_2) & (c_1, c_3) to construct

$W = \{c_2, c_1\}$ winner as $1 < 2$.

PART B

$W = C \iff$ No candidate is condorcet loser $\wedge$ all scores are the same.

$\Rightarrow$ Assume $W = C$; For any $c_i \in W$ they won a pairwise battle in P.

$\boxed{\text{Condorcet}}$ As $W = C$; every player wins at least one pairwise battle.

$\boxed{\text{Scores not =}}$ Assume scores not equal. Then $T_1 = \{c_1\}$ w/log and $T_2 = C \setminus \{c_1\}$ as $W \subseteq \underbrace{\tilde{P}}_{\text{The players in the pairs}} \subseteq C$ $\boxed{\therefore P = C}$

$P = \{\{c_1, c_k\} : 2 \leq k \leq \underbrace{m}_{\text{total players}}\}$

The pair $\{c_1, c_k\}$ is the only time c_k shows up in P.

\# As $c_k \in W$ (by assumption), it wins the head to head with 1 by a strict majority.

c_k picked generally. C_1 loses all head to heads. $\therefore c_1 \notin W$ and

$C \neq W$ contradiction!

$\Leftarrow$ Assume no condorcet loser + Plurality scores equal.

$\therefore P = \{(c_i, c_j) : i \neq j\}$

No loser implies every player wins
at least one head to head.

$\therefore C = W$