

Problem 1a

$$G = [13; 7, 7, 9, 3]$$

$$2\phi_7 + 2\phi_9 = 1$$

so we can just compute the shapley value for any one player.

I only need to consider winning permutations where 7

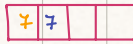
Pick player 7.

adds value to the game. 7 adds value when ① 7 is third



3 slots 3 options: $3!$ ways

② 7 is second but preceded by 7



$2!$ ways to arrange the 3's.

$$\phi_7 = (3! + 2!) / 4! = 1/3$$

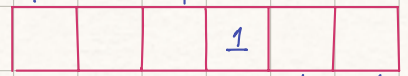
$$\phi_9 = (1 - 2\phi_7) / 2 = 1/6$$

Need 3 3's here out of 4 $\therefore \binom{4}{3} 3!$

Problem 1b

$$G = [10; 3, 3, 3, 3, 1, 1]$$

1 must be fourth to be influential.



$$\phi_1 = \binom{4}{3} \cdot 3! \cdot 2! / 6! = 1/15$$

$$\phi_3 = 13/60$$

2 ways of ordering 3 and 1.

Problem 2

Given a weighted voting game $G = [q; w_1, \dots, w_n]$

Prove that $w_i \leq w_j \Rightarrow \phi_i(G) \leq \phi_j(G)$

Let π be a permutation for which player i is pivotal.

Let $\pi_{i \leftrightarrow j}$ be the permutation obtained by swapping i and j .

Let S be the predecessors of i in π



As i is pivotal we have $w(S) < q$ but $w(S \cup \{i\}) \geq q$.

Now imagine $j \notin S$. Then $w(S \cup \{j\}) \geq w(S \cup \{i\}) \geq q$

$j \in S$ Then $w(S \setminus \{j\} \cup \{i\}) = w(S) - w_j + w_i < q$

Adding j to $S \setminus \{j\} \cup \{i\}$ is pivotal. $\therefore j$ is pivotal to S .

Observe that $\pi \neq \pi'$; then $\pi_{i \leftrightarrow j} \neq \pi'_{i \leftrightarrow j}$; and j will be pivotal for the set S' that are predecessors of i in π' (where i is pivotal)

Therefore the set of permutations where j is pivotal is at least as large

as the set of permutations i is pivotal in. i.e. $\phi(i) \leq \phi(j)$

Problem 4

Can you compute Banzhaf index in poly time? What about pseudopoly time?

No! To set Banzhaf index of null player i equal to $\phi_i = 0$, we need to check if a player is not null in a weighted voting game, which we know is NP-hard.

So unless $P = NP$; it is not possible to compute the Banzhaf index in poly time in the size of the instance.

Pseudopoly time? Yes \rightarrow Use dynamic programming.

$$\beta_i(G) = \frac{1}{2^{n-1}} \sum_{w=q-w_i}^{q-1} Z(w)$$

\rightarrow number of subsets of $[n]$ that have weight w !

Define $U(w,k) = \left| \left\{ S \subseteq \{1,2,\dots,k\} : \sum_{x \in S} w_x = w \right\} \right|$

number of subsets of $[k]$ that have value w

$Z(w) = U(w, n-1)$; Recursive Equation: $U(w,k) = U(w, k-1) + U(w-w_k, k-1)$; $U(w,1) = 1$ if $w = w_1$ or $w = 0$ otherwise } Base case

↑ Player k has no impact ↑ player k gets the subset upto k.

Problem 7 Compute the Schultz winner in poly time?

Suffices to compute $p[a,b]$ for all $a,b \in C$.

To compute $p[a,b]$

shortest paths & majority graph

Iterate k in $[\frac{1}{2}, \frac{1}{2}+1, \dots, n]$ // could do with binary search

* construct k -majority graph

* output largest k for which there is a path from a to b .

do it for all pairs $a,b \in C$ and then find a st $p[a,b] \geq p[b,a]$

* path checking $O(m)$

* search $O(n)$ or $O(\log n)$ → Total complexity $O(m^3 \log n)$

* # pairs $O(m^2)$

Problem 3 Game $G = (N, v)$; show that v can be expressed as a linear combination of unanimity games.

$$v = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) u_S$$

Fix some coalition Q . For $S \subseteq N$; $u_S(Q) = 1$ if $S \subseteq Q$ 0 otherwise.

Want to show:

$$v(Q) = \sum_{\substack{S \subseteq Q \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) u_S(Q)$$

$$\sum_{\substack{S \subseteq Q \\ S \neq \emptyset}} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot v(T) \right) \cdot 1$$

Re-write as

$$\sum_{T \subseteq Q} \sum_{S: T \subseteq S \subseteq Q} \dots$$

CASE I:

If $T=Q$, then $S=T=Q$.

$$(-1)^0 v(Q) = v(Q)$$

CASE II:

If $T \subset Q$, let $|T|=t$ $|Q|=q$; we have $q-t$ elements in Q that are not in T .

We have $\binom{q-t}{s}$ ways of making a set S of size $t+s$. s.t $T \subseteq S$ and $T \subset Q$. $\therefore |S|-|T|=s$; T contributes

Now s goes from $0, 1, \dots, (q-t)$ (If s were any bigger T would contain some elements of Q)

$$\therefore T's \text{ total contribution is } \sum_{s=0}^{q-t} \binom{q-t}{s} (-1)^s v(T)$$

$$= (1 + (-1))^{q-t} = 0$$

Taylor series / Binomial theorem / whatever you call this by

$$\binom{q-t}{s} (-1)^s v(T)$$

when $|S|=t+s$.

PROBLEM 5

Show that the Borda count winner cannot be a Condorcet loser.

We have m candidates $C = \{1, \dots, m\}$ and n voters.

Total score for all candidates $n \cdot (1+2+\dots+m-1) = n \cdot \frac{m(m-1)}{2}$. By the pigeon hole principle the winners score must be $\geq \frac{n(m-1)}{2}$ ##

$\hookrightarrow n$ voters vote once for every position.

FACT: For a fixed voter the Borda count of a candidate is the number of candidates ranked lower than the candidate.

$$\therefore \text{Borda score of candidate } b = \sum_{i \in [n]} \sum_{a \in C \setminus \{b\}} \mathbb{1}[a \text{ is ranked lower than } b]$$

Indicator fn.

$$= \sum_{a \in C \setminus \{b\}} \left[\sum_{i \in [n]} \mathbb{1}[a \text{ is ranked lower than } b] \right]$$

\hookrightarrow Number of times b beats a in head to head battles

Assume b is a Condorcet loser. b loses every head to head game i.e. the total votes b gets is $\leq \frac{n(m-1)}{2}$ (call $m-1$ one of them)

which contradicts ##

Problem 6

	num votes					
$s_1 = 3$	3	c_1	\succeq	c_2	\succeq	c_3
$s_2 = 2$	2	c_2	\succeq	c_1	\succeq	c_3
$s_3 = 2$	2	c_3	\succeq	c_2	\succeq	c_1

$$T_1 = \{c_1\} \quad T_2 = \{c_2, c_3\}$$

EARLY: Pit c_1 and c_2 : c_2 wins

$$P = \{ \{c_1, c_2\}, \{c_1, c_3\} \}$$

LATE: Pit (c_1, c_2) & (c_1, c_3) to construct

$$W = \{c_2, c_3\} \rightarrow \text{winner as } 1 < 2.$$

PART B

$W = C \iff$ No candidate is Condorcet loser \wedge all scores are the same.

\Rightarrow Assume $W = C$; For any $c_i \in W$ they won a pairwise battle in P .

Condorcet As $W = C$; every player wins at least one pairwise battle.

Score not Assume scores not equal. Then $T_1 = \{c_1\}$ and $T_2 = C \setminus \{c_1\}$ as $W \subseteq \tilde{P} \subseteq C \therefore P = C$

$$P = \{ \{c_1, c_k\} : 2 \leq k \leq m \}$$

\downarrow total players

\downarrow
The players in the pairs

The pair $\{c_1, c_k\}$ is the only time c_k shows up in P .

As $c_k \in W$ (by assumption), it wins the head to head with 1 by a strict majority.

c_k picked generally. c_1 loses all head to heads. $\therefore c_1 \notin W$ and

$C \neq W$ contradiction!

← Assume no Condorcet loser + Plurality scores equal.

$$\therefore P = \{(e_i, e_j) : i \neq j\}$$

No loser implies every player wins
at least one head to head.

$$\therefore C = W$$