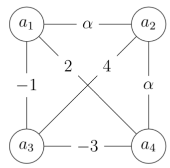
**Algorithmic Foundations of Collective Decision Making: Tutorial 2**

1. Consider the symmetric additively separable hedonic game $(N, u)$ with $N = \{a_i : 1 \leq i \leq 4\}$ and utilities given by the figure below, where $\alpha \in \mathbb{R}$ is a parameter.




Consider the coalition structure $\mathfrak{C} = \{\{a_1, a_2, a_3\}, \{a_4\}\}$. Determine for which values of $\alpha$ the coalition structure $\mathfrak{C}$ is

- individually rational,
- Nash stable,

Let $\pi = \{ \{a_1, a_2, a_3\}, \{a_4\}\}$

$u_1(\pi) = u_1(2) + u_2(3) = a - 1$

$u_2(\pi) = u_2(1) + u_2(3) = 4 + a$

$u_3(\pi) = u_3(1) + u_3(2) = 3$

$u_4(\pi) = 0$

Player doesn't want to be alone

For $\pi$ to individually rational we need.

$a - 1 \geq 0$ So 1 is at least as good as being alone

$a \geq 1$.

↳ This also handles 2's case which needed $a \geq -4$

So we have $\boxed{\alpha \geq 1}$ for $\pi$ to be IR.

For $\pi$ to be nash stable we want to stop $a_1$ from joining $a_4$; which would give $a_1$ a utility of 2.

$\therefore 2 \leq a - 1$ or $\boxed{\alpha \geq 3}$ *

But to stop $a_4$ from joining the rest of them we need

$a - 1 < 0$ or $\boxed{\alpha < 1}$ #

utility $a_4$ gets from joining $\{a_1, a_2, a_3\}$

Both # and * not possible.

There is No $\alpha$ for which $\pi$ is Nash Stable.

For $\pi$ to be in the core we need to rule out blocking coalitions.

$\{a_1, a_4\}$ is blocking for $\alpha < 3$. To see why

$a_1$'s utility under $\pi$

$a_1$'s utility under: $2 > \alpha - 1 \Rightarrow$ ($\alpha \geq 3$) need to be non blocking

$a_4$'s utility ": $2 > 0$ — $a_4$'s utility under $\pi$

$\{a_1, a_2, a_4\}$ is blocking for $\alpha > 4$

$a_2$'s old ulitily

$a_2$'s new ulity: $2\alpha > 4 + \alpha$.

$\Rightarrow$ need $\alpha \leq 4$ to be non blocking

$\boxed{3 \leq \alpha \leq 4} \Rightarrow \pi$ is in the core

2. Prove the following theorem stated in the lecture: Every symmetric ASHG admits a Nash stable coalition structure.

Let $\pi^*$ be the coalition structure that maximises social welfare. Then we will show that $\pi^*$ is nash stable.

$\sum_{i \in N} u_i(\pi) = SW(\pi)$

Let $\pi'$ be the coalition structure formed by player $i$ moving.

Let $\pi^* = \{S_1, \dots S_m\}$. without loss of generality assume
$i \in S_1$ in $\pi^*$
$i \in S_2$ in $\pi'$

utility $i$ gets from being in $S_1$

players in $S_1$ that are not $i$, what utility do they get

$$SW(\pi) = u_i(S_1) + \sum_{j \in S_1 \setminus \{i\}} u_j(S_1) + \sum_{j \in S_2} u_j(S_2) + \square$$

remaining terms for $S_3 \dots S_m$

$\square$ is the same in both equations

$$SW(\pi') = u_i(S_2 \cup \{i\}) + \sum_{j \in S_1 \setminus \{i\}} u_j(S_1 \setminus \{i\}) \; \# + \sum_{j \in S_2} u_j(S_2 \cup \{i\}) \; \# + \square$$

Now $u_j(S_1) = u_j(i) + u_j(S_1 \setminus \{i\})$ for any $j \in S_1 \setminus \{i\}$ → plug into #

$u_j(S_2 \cup \{i\}) = u_j(i) + u_j(S_2)$ " $j \in S_2$ → plug into #

$$\therefore \underbrace{SW(\pi^*) - SW(\pi')}_{>0} = \left[u_i(S_1) - u_i(S_2 \cup \{i\})\right] + \left[\sum_{j \in S_1 \setminus \{i\}} u_j(i)\right] - \left[\sum_{j \in S_2} u_j(i)\right]$$

$\sum_{j \in S_1 \setminus \{i\}} u_j(i) = u_i(S_1)$, $\sum_{j \in S_2} u_j(i) = u_i(S_2 \cup \{i\})$

As $u_i(j) = u_j(i)$ By symmetry (these follow).

as $\pi^*$ maximizes social welfare

$$= 2\left[u_i(S_1) - u_i(S_2 \cup \{i\})\right]$$

$i$ switched because $u_i(S_2 \cup \{i\}) > u_i(S_1)$

so this is 2* negative number.

CONTRADICTION!

③

3. Construct a symmetric additively separable hedonic game whose core is empty.

Think of what this statement really means?

" I can come up with a network (undirected) such that no one ever wants to stay in any relationship "

First relax this need for the undirected / symmetry. Is this problem then easy?

What comes to your mind when you read the bit in blue?

This game shows up in hollywood movies, novels, college campuses. It's everywhere!

Answer : A Love Triangle.

We have Alice, Bob, Charlie

① They resent the idea of all being together in a 3 way marriage. They'd rather be alone!

② But they'd rather be with someone than alone.

③ They have a cyclic love!

Alice loves Bob more than Charlie

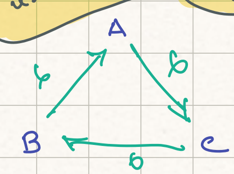
Bob loves Charlie more than Alice

Charlie loves Alice more than Bob!

↓ Expressed just a hedonic game ; this works.

→ But fails for additive games.

Also makes use of directed edges :(




Grand coalition is in core :(

BC blocked by AB

AB blocked by AC

AC blocked by BC.

That's my hollywood script.

Make it into a graph

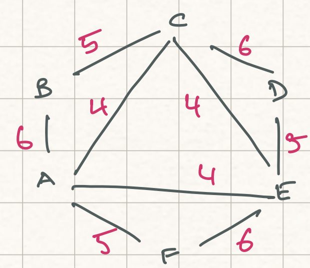
Now this doesn't work!

I had the right idea but this graph is too rigid.

Not enough ways to label things

## Lift up! Add more players.

Make new world when
→ 4 man relationship is AWFUL
→ 3 man better than 2 better than 1
→ Use love △ idea




Observe : No one wants to be alone.

Any coalition of 2 is blocked by some coalition of size

- AB, BC blocked by ABC
- CD, DE " CDE
- AF, FE " AFE
- AC, CE, AE " ACE

**The core is Empty**

No coalition of 3 is in core

ABC blocked by CED

* c gets 9 — c gets 10
* DE get −∞ — D, E get 10

Similarly,

CED blocked AFE
↓ better off

AFE blocked by ABC.

- No coalition ≥ 4 is in core!

4. Consider an instance $\mathcal{I}$ of the stable matching problem such that when we execute the Gale–Shapley algorithm on $\mathcal{I}$ with men as the proposing side and when we execute it with women as the proposing side, we get the same matching, which we will denote by $S$.

Prove or disprove the following statements:

(a) There exists a unique stable matching for $\mathcal{I}$.

(b) At least one person receives their top choice in $S$.

## (a)

The first claim is true!

Let S be the output of Gale Shapley regardless of which group proposes.

Let $(x, w) \in S$. Then $x$ is $w$'s best and worst valid partner.
Likewise $w$ is $x$'s " " .

( Since S is the output regardless of men proposing or women proposing)

If you have more than 1 stable matching,
Then there cannot be a pair that is both best and worst!

For example: Assume $S_2 \neq S$.

$(x, w) \in S$ — if $w$ is best partner, then $w'$ will be worst.

$(x, w') \in S_2$ (Another stable matching) — but $(x, w') \notin S$ when men are rejectors.

contradiction!

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## (b)

1. Fix desired outcome
2. Play one side making sure each player gets knocked out once

| | | | | |
|---|---|---|---|---|
| A | X | W | Y | FREE |
| B | Y | X | FREE | |
| C | X | Y | | Z |
| D | Y | Z | X | W |

| | | | | |
|---|---|---|---|---|
| W | D | A | FREE | |
| X | D | B | C | A |
| Y | A | C | D | B |
| Z | C | D | FREE | |

1. C needs to knock A;
2. D needs to knock B
3. B knocks C
4. C knocks D

(b) Try and set each player so they get II pref

A–D proposing: ~~AX~~, ~~BY~~, ~~CX~~, ~~DY~~, AW, BX, CY, DZ

W–Z proposing: ~~WD~~, ~~XD~~, ~~YA~~, ~~ZC~~, WA, YC, ZD, XB

- X knocks W.
- W knocks Y
- Y knocks Z
- Z knocks X

5. Suppose the Gale–Shapley algorithm outputs a matching $S$ on instance $\mathcal{I}$. Consider a subset of men $M' \subseteq M$, and modify $\mathcal{I}$ so that for each man $m \in M'$ his top-ranked woman in the new instance is the woman he is matched to in $S$ (and otherwise their rankings over women are arbitrary); the preferences of the men in $M \setminus M'$ remain unchanged. Denote the modified instance by $\mathcal{I}'$. Argue that the Gale–Shapley algorithm outpus $S$ on $\mathcal{I}'$ as well.

$\mathcal{I}$

| M | | | |
|---|---|---|---|
| X | C | A | B |
| Y | C | A | B |
| Z | A | C | B |

| F | | | |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | Z | Y | X |

Hint: Try knocking everyone off their first preference! Create cycles!

$M' = \{X\}$

Output S on $\mathcal{I}$

XC, YC, ZA, XA, [ZC, YA, XB] S

Output S' on $\mathcal{I}'$

$\mathcal{I}'$

| | | | |
|---|---|---|---|
| X | B | A | C |
| Y | C | A | B |
| Z | A | C | B |

Women remain the same!

[XB, YC, ZA] S'

$S \neq S'$