1. Prove that deciding if a given player in a weighted voting game is a null player in coNP-complete.

We are given a weighted voting game $G = [(w_1, \dots w_n), k]$ and some $i \in [n]$ and asked to show that the decision problem

$\boxed{\text{is } i \text{ a dummy player in } G}$ #

is CO-NP complete.

Alternatively, we could consider the complement of this decision problem

##

is $\boxed{i \text{ NOT a dummy player in } G}$ and show that this decision problem is NP complete.

Player i is NOT dummy if $\exists S \subseteq [n] \setminus \{i\}$

S is losing

$$\sum_{x \in S} w_x < \sum_{x \in S} w_x + w_i \quad \text{—} *$$

$S \cup \{i\}$ is winning

If someone gave us S (think of S as a witness); then we can check * in linear time in n.

So we have a short proof. ## in NP.

To show ## is NP complete we need to show that any problem $x \in NP$, karp reduces to ##.

KARP Reduction

We show that PARTITION $\leq_K$ Checking if i is NOT dummy in G.

(known to be NP complete)

Given an instance of PARTITION

$(a_1 \dots a_n)$ and parameter K

we set up a weighted voting game

$w_i = a_i \quad \forall i \in [n]$ and set the quota of

$w_{n+1} = 1$ the game to be $k+1$

This transformation is clearly poly(n) —

Now to finish the reduction we want to show

$$\underbrace{(a_1, \dots a_n, K)}_{x_P} \in \text{PARTITION} \iff$$ Player n+1 not dumy in transformed game.

$\Rightarrow$ Assume

$x_P \in$ PARTITION

$\exists S \subset N$ st

$$\sum_{x \in S} a_x = K$$

Then $\underbrace{\sum_{x \in S} w_x}_{S \text{ is Losing}} < \underbrace{\sum_{x \in S} w + w_{n+1}}_{S \cup \{n+1\} \text{ winning}}$ | n+1 is not dummy.

$\Leftarrow$ Assume n+1 is not dummy

Then $\exists S \subset N$ s.t

S is losing but $S \cup \{n+1\}$ is winning

$\therefore \sum_{x \in S} a_x = k$ as adding 1 to it makes winning but removing 1 makes it losing.

This is a partition

2. The VERTEX COVER problem is given by an undirected graph $G = (V, E)$ and a positive integer k. A pair (G, k) is a yes-instance if G admits a vertex cover of size k, i.e., a subset of vertices $S \subseteq V$ with $|S| = k$ such that for each edge $\{u, v\} \in E$ we have $u \in S$ or $v \in S$.

   Consider the following mapping from an instance of VERTEX COVER to a vector weighted voting game. Given an instance with n vertices and m edges, we construct a game with n players $1, \ldots, n$ that is a conjunction of m weighted voting games, one per edge. For each edge $e = \{u, v\}$ the game G^e has quota 1; the weights of players u and v are 1, and the weights of all other players are 0. Use this construction to prove NP-hardness of the following decision problem (you may need to modify the basic construction to do this, e.g., by adding extra games or changing weights).

   Given a vector weighted voting game that is a conjunction of t weighted voting games $G^1 \wedge \cdots \wedge G^t$, is the game G^t *relevant*, i.e., is it the case that $G^1 \wedge \cdots \wedge G^{t-1}$ and $G^1 \wedge \cdots \wedge G^t$ do not have the same set of winning coalitions?

conjuctive notation :

G_i : weighted voting game.

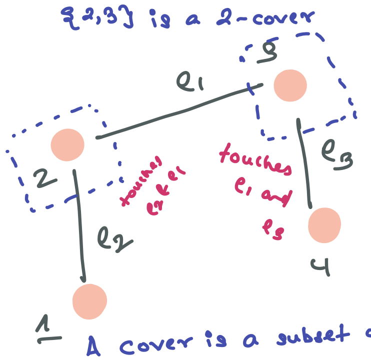
Let $G = G_1 \wedge G_2 \wedge \dots G_t$ be a game described by the conjuction of t games.

By conjunction, I mean, if S wins in $G \Rightarrow S$ wins for ALL G_i.

You are told to show that the following decision problem is NP-HARD.

Aside: k-vertex cover

$\{2,3\}$ is a 2-cover



A cover is a subset of nodes of the graph s.t I touch all edges

Given $G = G_1 \wedge \dots G_t$ is G_t relevant?

"A game is relevant if adding G_t to G changes the set of winning coalitions"

Known to be NP-HARD

Want to show VertexCOVER(G_r, k) $\leq_K$

For $e \in E$ and $e = (u,v)$ construct a game G^e with $|V|$ players where any coalition that contains u or v wins. i.e quota $= 1$

We are given

$G_r = (V, E)$
$V = [n]$
$|E| = m$
$k < n$

Let $\tilde{G} = G^{e_1} \wedge \dots G^{e_m}$

Clearly this transformation is poly(n).

Now define G^* as a weighted voting game with $|V|$ players where $w_i = 1$ $\forall i \in [n]$; and quota $\underbrace{k+1}$

from the vertex cover prob

Want to show if

$(Gr, k) \in$ Vertex Cover $\iff$ G^* is relevant to $\tilde{G}$

$\Rightarrow$ Let S be a vertex cover of Gr of size k.
Then the coalition S loses in G^* as the quota is $k+1$ and $|S| = k$.

But S wins $\tilde{G}$ as S has a node that touches every edge (by vertex cover property); so for every game G^e it will win as $u \in S$ s.t $e = (*, u)$ or $(u, *)$.

$\Leftarrow$ G^* is relevant. $\exists$ coalition S s.t S is winning in $\tilde{G}$ but not G^*. [By relevance]
For S to win $\tilde{G}$; it must be a cover of Gr of size at most k.

as it's losing in G^*

3. Consider two simple games $G^1 = (A, v^1)$ and $G^2 = (A, v^2)$ with the same set of players A. Suppose that a player $i \in A$ is not a null player in both games. Can we conclude that i is not a null player in the game $G^\cap = (A, v^\cap)$, with the characteristic function $v^\cap$ given by $v^\cap(C) = \min\{v^1(C), v^2(C)\}$? What about the game $G^\cup = (A, v^\cup)$, where $v^\cup$ is given by $v^\cup(C) = \max\{v^1(C), v^2(C)\}$?

A few ways to tackle this. I'm writing what I think is the simplest.

$A = \{1,2,3\}$

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	
v^1	0	0	0	0	$\underline{1}$	0	1	1	
v^2	0	0	0	0	0	$\underline{1}$	1	1	
$v^\cap$	0	0	0	0	0	0	1	1	{1 is null}

— : Adding 1 to {2} changes value. 1 is NOT DUMMY in v^1

— : Adding 1 to {3} " " . 1 is NOT DUMMY in v^2.

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	
v^1	0	0	1	0	1	$\underline{1}$	1	1	
v^2	0	0	0	1	$\underline{1}$	1	1	1	
$v^\cap$	0	0	0	0	0	0	1	1	{1 is null}

— : Adding 1 to {3} changes value. 1 is NOT DUMMY in v^1

— : Adding 1 to {2} " " . 1 is NOT DUMMY in v^2.

4. Prove that any outcome in the core maximizes the social welfare, i.e., for any coalitional game G it holds that if $(CS, \mathbf{x})$ is in the core of $G = (N, v)$ then for any coalition structure CS' for G we have $\sum_{C \in CS} v(C) \geq \sum_{C' \in CS'} v(C')$.

Assume that the statement is false.

$$\exists\ \tilde{CS} \text{ s.t } \sum_{S \in CS} v(S) < \sum_{\tilde{S} \in \tilde{CS}} v(\tilde{S})$$

$$\gamma = \sum_{\tilde{S} \in \tilde{CS}} x(\tilde{S}) \overset{①}{=} \sum_{i \in [n]} x_i \overset{②}{=} \sum_{S \in CS} v(S) \underset{③}{<} \sum_{\tilde{S} \in \tilde{CS}} v(\tilde{S}) \underset{④}{\leq} \sum_{\tilde{S} \in \tilde{CS}} x(\tilde{S}) = \gamma$$

① $\tilde{CS}$ is a partition of $[n]$

② As $x \in \text{core}(G)$; it must be efficient! Called an outcome in lecture slides!

③ By assumption

④ By definition of core $x(S) \geq v(S)\ \forall S \subseteq [n]$

But now $\gamma < \gamma$? Contradiction!

5. Suppose an outcome $(CS, \mathbf{x})$ is in the core of $G = (N, v)$. Show that for every other coalition structure CS' with $\sum_{C \in CS} v(C) = \sum_{C' \in CS'} v(C')$ there is a payoff vector $\mathbf{y}$ such that $(CS', \mathbf{y})$ is in the core of G.

Set $y = x$.

Want to show that $x(S) = v(S) \quad \forall S \in CS'$ [Then (CS', x) is a valid outcome

Note: Already have $x(S) \geq v(S) \quad \forall S \subseteq [n]$

So just is enough ?

$$\boxed{\sum_{S' \in CS'} x(S') \overset{1}{=} \sum_{i \in [n]} x_i \overset{2}{=} \sum_{S \in CS} v(C) \overset{3}{=} \sum_{S \in CS'} v(S) \quad **}$$

1. CS' is a partition of $[n]$
2. (CS, x) is a valid outcome
3. Problem statement

Assume $\exists \tilde{S} \in CS'$. st

$$x(\tilde{S}) > v(\tilde{S})$$

$$\therefore \sum_{S' \in CS'} x(S') > \sum_{S \in CS'} v(S)$$

This contradicts $**$

6. In class, we proved that a superadditive simple game has a non-empty core if and only if it has a veto player. We also claimed that the following corollary holds: a payoff vector $(x_1, \ldots, x_n)$ is in the core of a superadditive simple game $G = (N, v)$ if and only if $x_i = 0$ for each player i who is not a veto player. Prove this corollary.

Want to prove $\vec{x} \in \text{core}(G) \iff x_i = 0$ if i not veto player.

$\Rightarrow$ Assume $\vec{x} \in \text{core}(G)$.

$\boxed{\text{Notation: } N_{-i} = N \setminus \{i\}}$

Let i be a non veto player. Then $\exists\, S \subseteq N_{-i}$ s.t $v(S) = \underline{1}$

As $v(S) = 1$; by monotonicity $v(N) = \underline{1}$. But as $x \in \text{core}(G)$ $x(N) = \underline{1}$.

$$\underset{1}{\underline{x(N)}} = \underset{1}{\underline{\sum_{j \in S} x(j)}} + \sum_{j \notin S} x(j)$$

must be 0

$\therefore$ $x_i = 0$ as sum is 0 as $\underline{i \notin S}$

$\Leftarrow$ Assume $x_i = 0$ $\forall i$ non veto.

Want to show $x(S) \geq v(S)$ $\forall S \subseteq [n]$

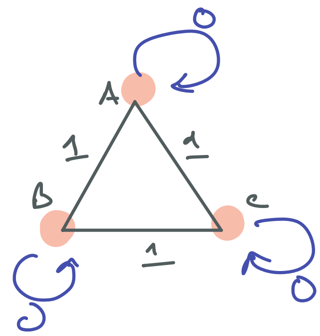
- If S is a losing coalition then $v(S) = 0$; so $x(S) \geq v(S)$ as $x(S) \in \{0, 1\}$

- Let S is be a winning coalition with all veto players.
(only way to win is have all veto players)

As $x(N) = 1$ $\quad \underbrace{x(N)}_{1} = \underbrace{x(S)}_{=1} + \underbrace{x(S^c)}_{0 \text{ by assumption}}$ complement

$= v(S) \geq v(S)$.

7. Consider a 3-player simple game where a coalition is winning if and only if it contains at least 2 players. Can this game be represented as an induced subgraph game? Assume that self-loops are allowed.



But now $v(\{A,B,C\}) = 3$

Not allowed!

if I make $v\{A,B,C\} = 1$

Then $v(\{A,B\}) = 1/3$

not allowed!

For every $n \geq 1$, construct an n-player convex simple game where every two players are symmetric. How many such games are there for each value of n? Justify your answer.

There are 2 games for every n.

G_n^1 : All $S \subseteq N : v(S) = 0$

Easy to show the game is symmetric and convex. All valuations are 0.

G_n^2 : All $S \subset N$ $v(S) = 0$

$v(N) = 1$

The only way to score a point is be in the grand coalition.

So once again symmetric + convex!

Claim: There are no more symmetric simple convex game!

Assume w/log $n > 1$

Let C be the smallest winning coalition of a game G that is not G_n^1 & G_n^2

Pick $j \notin C$ and $i \in C$.

By symmetry, $(C \setminus \{i\}) \cup \{j\}$ is also a winning coalition.

As C is the $\boxed{\text{smallest}}$ winning coalition

$v(C \cup \{j\}) = v(C)$ // monotonicity

But $C \setminus \{i\}$ — is not winning! // $C \setminus \{i\} \subset C$ [And C is smallest winning] j positively impacts $C \setminus \{i\}$ but j does not impact C.

C is bigger than $C \setminus \{i\}$. Breaks convexity! which requires

$$v(S \cup \{j\}) - v(S) \geq v(S' \cup \{j\}) - v(S') \quad \forall j \in [n]$$

if $S' \subseteq S$